PRACTICAL AND THEORETICAL

GEOMETRY

PART II

McDOUGALL

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PART II

THEORETICAL

BY

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PREFACE.

Part II. completes the course in Geometry prescribed by the Department of Education of Ontario for the Lower and Middle Schools of High Schools and Collegiate Institutes.

The exercises, which have been carefully selected and graded, will be found suitable for the work of average classes and just about sufficient in number to fix the subject-matter of the fundamental propositions in the minds of the pupils.

It is important that the accuracy in drawing required in Part I. should be continually practised. From time to time the practical exercises of Part I. may with advantage be reviewed in connection with the more purely theoretical methods of Part II.

The algebraic treatment of proportion is openly used and the correlation of this important part of the work in Geometry with the previous knowledge of the pupils is thereby established.

While the method of limits has been introduced in connection with tangents, the older proofs have been indicated in the exercises and may be substituted for those in the text by teachers who prefer to do so.

Special care has been taken to avoid redundancy and obscurity of language.

Acknowledgment and thanks are due to J. C. Glashan, LL.D., F.R.S.C., for many valuable suggestions and corrections, and also for suggestive criticism to Professor H. M. Tory, M.A., D.Sc., of McGill University ; I. T. Norris, B.A., Mathematical Master of the Ottawa C.I. ; I. J. Birchard, M.A., Ph.D., of Jameson Avenue C.I., Toronto, and other experienced educationists.

SYMBOLS AND ABBREVIATIONS.

The following symbols and abbreviations are used :—

e.g.	*exempli gratia*, for example.
i.e.	*id est*, that is.
$\because$	because, since.
$\therefore$	therefore.
rt.	right.
st.	straight.
$\angle$, $\angle$s, $\angle$d	angle, angles, angled.
$\triangle$, $\triangle$s	triangle, triangles.
$\parallel$, $\parallel$s	parallel, parallels.
$\parallel$gm, $\parallel$gms	parallelogram, parallelograms.
4-gon	quadrilateral.
sq., sqs.	square, squares.
AB^2	the square on AB.
$+$	plus, together with.
$-$	minus, diminished by.
AB.CD	the rectangle contained by AB and CD.
AB : CD, or $\frac{AB}{CD}$	the ratio of AB to CD.
$\perp$	is perpendicular to, a perpendicular.
$=$	is equal to, equals.
$>$	is greater than.
$<$	is less than.

CONTENTS.

PART II.

INTRODUCTION.

1. Any combination of points, lines, surfaces and solids is called a **figure**.

2. **Geometry** is the science which investigates the properties of figures and the relations of figures to one another.

3. A surface has been defined (Part I.) as that which has length and breadth, but no thickness. A surface is thus the boundary between two parts of space.

A line is that which has length, but neither breadth nor thickness; the boundary between two parts of a surface is consequently a line.

4. The following property distinguishes straight lines from curved lines and may be used as the definition of a straight line :—

Two straight lines cannot have two common points without coinciding altogether.

This is sometimes stated as follows :—Joining two points there is always one and only one straight line.

It follows from this definition that two straight lines cannot enclose a space.

Can the circumferences of two equal circles coincide in two points without coinciding altogether?

5. The following property distinguishes plane surfaces from curved surfaces and may be used as the definition of a plane surface:—

The straight line joining any two points on a plane surface lies wholly on that surface.

Give examples of curved surfaces on which straight lines may be drawn in certain directions. Notice the force of the word "any" in the definition above.

6. In **Plane Geometry** the figure, or figures, considered in each proposition are confined to one plane, while **Solid Geometry** treats of figures the parts of which are not all in the same plane.

Plane Geometry is also called Geometry of Two Dimensions (length and breadth), and Solid Geometry is called Geometry of Three Dimensions (length, breadth and thickness).

GEOMETRICAL REASONING.

7. Two methods of investigation which may be distinguished as the Practical Method and the Theoretical Method were used in Part I.

Some properties were simply tested by measurement, paper-folding, etc., while in other cases it was shown that the property followed as a necessary result from others that were already known to be true.

This second, or Theoretical Method, has certain advantages over the first, or Practical Method. Measurements, etc., are never exact, and in many cases cannot be made directly; but in the Theoretical Method, starting from certain simple statements, called **axioms**, the truth of which is assumed, the consequent statements follow with absolute certainty.

The Practical Method is also known as the Inductive Method of Reasoning, and the Theoretical Method as the Deductive Method.

8. In Part I., figures were found to be equal by making a tracing of one of them and fitting the tracing on the other. In many cases the process may be made a mental operation and the equality shown with absolute certainty by means of the following axiom:—

A figure may be mentally transferred from one position to another without change of form or size.

9. When two figures are shown to be exactly equal in all respects by supposing one to be made to fit exactly on the other the proof is said to be by the **method of superposition.**

10. In general a **proposition** is anything stated or affirmed for discussion.

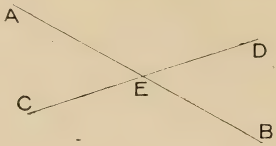
In mathematics a **proposition** is a statement of either a truth to be demonstrated or of an operation to be performed. It is called a **theorem** when it is something to be proved, and a **problem** when it is an operation to be performed.

Example of Theorem:—If two straight lines cut one another, the vertically opposite angles are equal.

Example of Problem:—It is required to bisect a given line-segment.

11. Propositions are commonly stated in two ways:—First, the **General Enunciation,** in which the property is stated as true for all figures of a class, but without naming any particular figure, as in the two examples given in § 10; second, the **Particular Enunciation,** in which the proposition is stated to be true of the particular figure in a certain diagram.

Examples of Particular Enunciation :—1. Let AB and CD be two st. lines cutting at E.



It is required to show that ∠ AEC = ∠ BED, and that ∠ AED = ∠ BEC.

2. Let AB be a given line-segment.

It is required to bisect AB.

12. In general, the enunciation of a proposition consists of two parts: the hypothesis and the conclusion.

The **hypothesis** is the formal statement of the conditions that are supposed to exist, *e.g.*, in the first example of § 10, "If two straight lines cut each other."

The **conclusion** is that which is asserted to follow necessarily from the hypothesis, *e.g.*, "the vertically opposite angles are equal to each other."

Commonly, the hypothesis of a proposition is stated first, introduced by the word "if," and the two parts hypothesis and conclusion are separated by a comma. Sometimes, however, the two parts are not so clearly distinguished, *e.g.*, in the proposition :—The angles at the base of an isosceles triangle are equal to each other. In order to show the two parts, this statement may be changed as follows :—If a triangle has two sides equal to each other, the angles opposite these equal sides (or angles at the base) are equal to each other.

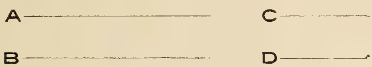
13. The demonstration of a theorem depends either on axioms or on other theorems that have been previously shown to be true.

The following are some of the axioms commonly used in geometrical reasoning :—

1. Things that are equal to the same thing are equal to each other.

If A = B, B = C, C = D, D = E and E = F, what about A and F ?

2. If equals be added to equals the sums are equal.



Thus if A, B, C, D be four line-segments such that A = B and C = D, then the sum of A and C = the sum of B and D.

3. If equals be taken from equals the remainders are equal.

Give example.

4. If equals be added to unequals the sums are unequal, the greater sum being obtained from the greater unequal.

Give example. Show also, by example, that if unequals be added to unequals the sums may be either equal or unequal.

5. If equals be taken from unequals the remainders are unequal, the greater remainder being obtained from the greater unequal.

6. Doubles of the same thing, or of equal things, are equal to each other.

7. Halves of the same thing, or of equal things, are equal to each other.

8. The whole is greater than its part, and equal to the sum of all its parts.

Give examples.

9. Magnitudes that coincide with one another, that is, which fill exactly the same space, are equal to each other.

These simple propositions, and others that are also plainly true, may be freely used in proving theorems.

14. The solutions of such simple problems as the following, which have been effected by practical methods in Part I., are assumed when necessary in proving theorems.

To bisect a given angle.

To bisect a given line-segment.

To draw a perpendicular to a given straight line from a point either within or without the line.

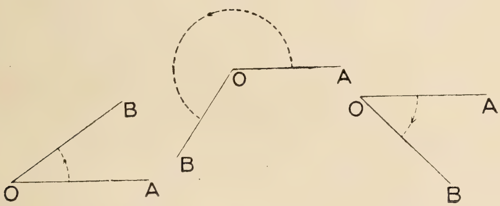
To make an angle equal to a given angle.

CHAPTER I.

Angles and Triangles.

15. **Definition** :—When two straight lines are drawn from a point the figure thus formed is called an **angle.** The point from which the lines are drawn is called the **vertex** of the angle, and the two straight lines are called the **arms** of the angle.

16. Suppose a straight line OB to be fixed, like a rigid rod on a pivot, at the point O, and be free to rotate in the plane of the paper.



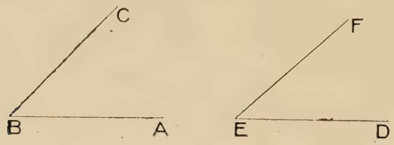
If the line OB start from any position OA, it may rotate in either of two directions—that in which the hands of a clock rotate, or in the opposite.

When OB starts from OA and stops at any position an angle is formed with O for its vertex and OA and OB for its arms.

17. An angle is said to be positive or negative according to the direction in which the line that traces out the angle is supposed to have rotated. The direction contrary to that in which the hands of a clock rotate is commonly taken as positive.

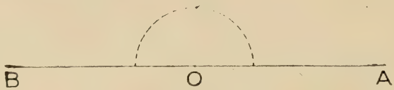
18. The magnitude of an angle depends altogether on the amount of rotation, and is quite independent of the lengths of its arms.

19. If we wish to compare two angles ABC and DEF we

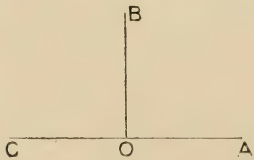


may suppose the angles ABC to be placed on DEF so that B falls on E and BA along ED. The position of BC with respect to EF will then show which of the angles is the greater and by how much it is greater than the other.

20. When the revolving line OB has made half of a complete revolution from the initial position OA the angle formed is a **straight angle.**



The arms of a straight angle are thus in the same straight line and extend in opposite directions from the vertex. At the point O, in the diagram, there are two straight angles on opposite sides of the straight line AOB, the two straight angles making up the complete revolution.



21. If a straight line, starting from OA, rotates in succession through two equal angles AOB, BOC, the sum of

which is a straight angle, each of these angles is called a **right angle.**

A right angle is thus one-half of a straight angle, or one-quarter of a complete revolution.

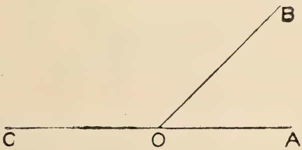
Each arm of a right angle is said to be **perpendicular** to the other arm.

22. Another way of giving the definitions of right angle and perpendicular is the following:—

If a straight line standing on another straight line makes the adjacent angles equal to each other, each of these angles is called a right angle; and the straight line which stands on the other is said to be perpendicular to it.

23. **Theorem:—The angles which one straight line makes with another on the same side of that other are together equal to two right angles.**

Let a st. line starting from OA revolve through two successive ∠s

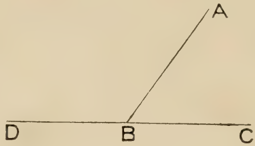


AOB, BOC such that OC is in the same st. line with OA, and consequently AOC is a st. ∠.

The sum of the ∠s AOB, BOC being a st. ∠, is equal to two rt. ∠s.

24. **Theorem:—If at a point in a straight line two other straight lines on the opposite sides of it are together equal to two right angles, the two straight lines are in the same straight line.**

At the point B in the st. line AB, let the two ∠s ABC, ABD, on opposite sides of AB, be together equal to two rt. ∠s.



It is required to prove that BD and BC are in the same st. line.

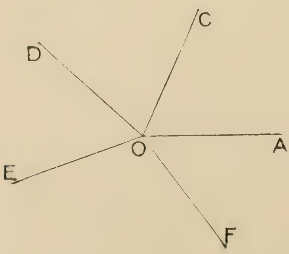
$$\angle\ ABD + \angle\ ABC = \text{two rt. } \angle s$$
$$= \text{a st. } \angle .$$

That is, DBC is a st. ∠, and

∴ DBC is a st. line.

25. **Theorem:—If any number of straight lines meet at a point, the sum of the successive angles is four right angles.**

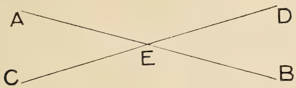
Let OB, starting from the position OA, and rotating in the positive direction, trace out the successive ∠s: AOC, COD, DOE, EOF, FOA.



The sum of the successive $\angle$s is a complete revolution, and therefore equal to four rt. $\angle$s.

26. **Theorem :—If two straight lines cut one another, the vertically opposite angles are equal to each other.**

Let the two st. lines AB, CD cut each other at E.



It is required to show that $\angle$ AEC = $\angle$ BED and that $\angle$ AED = $\angle$ CEB

$\because$ CED is a st. line,

$\angle$ AEC + $\angle$ AED = two rt. $\angle$s. (§ 23.)

$\because$ AEB is a st. line,

$\angle$ AED + $\angle$ DEB = two rt. $\angle$s.

$\therefore$ $\angle$ AEC + $\angle$ AED = $\angle$ AED + $\angle$ DEB.

From each of these equals take away the common $\angle$ AED and the remainders must be equal to each other.

$\therefore$ $\angle$ AEC = $\angle$ DEB.

In the same manner it may be shown that $\angle$ AED = $\angle$ CEB.

27. **Definition** :—When two angles are such that their sum is two right angles, they are said to be **supplementary** angles, or each angle is said to be the **supplement** of the other.

28. **Definition** :—When two angles are such that their sum is one right angle, they are said to be **complementary** angles, or each angle is said to be the **complement** of the other.

29.—Exercises.

1. If one of the four ∠s made by two intersecting st. lines be 17°, find the number of degrees in each of the other three.

2. If the interior ∠s of a polygon are equal to each other, the exterior ∠s made by producing the sides of the polygon are also equal to each other.

3. If two adjacent ∠s made by one st. line meeting another be bisected, the two bisectors are at rt. ∠s to each other.

4. E is a point between A and B in the st. line AB; DE, FE are drawn on opposite sides of AB and such that ∠DEA = ∠FEB. Show that DEF is a st. line.

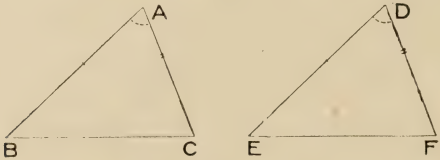
5. Four st. lines, OA, OB, OC, OD, are drawn in succession from the point O, and are such that ∠ AOB = ∠ COD and ∠ BOC = ∠DOA. Show that AOC is a st. line, and also that BOD is a st. line.

6. The bisectors of a pair of vertically opposite ∠s are in the same st. line.

First Case of the Equality of Triangles.

30. **Theorem:—If two triangles have two sides and the contained angle of one respectively equal to two sides and the contained angle of the other, the two triangles are congruent.**

Let ABC and DEF be two △s having AB = DE, AC = DF and ∠ A = ∠ D.



It is required to show BC = EF, ∠ B = ∠ E, ∠ C = ∠ F, and area of △ ABC = area of △ DEF.

Let $\triangle$ ABC be applied to $\triangle$ DEF so that point A falls on point D and AB falls along DE.

AB = DE, $\therefore$ point B must fall on point E;

$\angle$ A = $\angle$ D, $\therefore$ AC must fall along DF,

and consequently as AC = DF, the point C must fall on the point F;

$\therefore$ $\triangle$ ABC coincides with $\triangle$ DEF

and the two $\triangle$s are congruent.

31.—Exercises.

1. Draw the diagram and prove the proposition in § 30 when one $\triangle$ has to be supposed to be turned over before it can be made to coincide with the other.

2. Through A and B, the ends of a line-segment AB draw CAD and EBF both perpendicular to AB, and such that CA = AD and EB = BF. Prove CE = DF.

3. Two line-segments AOB, COD cut one another at O, so that OA = OB and OC = OD; join AD and BC, and prove $\triangle$s AOD, BOC congruent.

4. Prove that all chords of a circle which subtend equal angles at the centre are equal to each other.

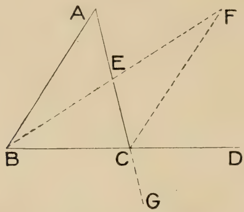
5. If with the same centre O, two circles be drawn, and st. lines ODB, OEC be drawn to meet the circumferences in D, E, B, C; prove that BE = DC.

6. ABCD is a quadrilateral having the opposite sides AB, CD equal and $\angle$ B = $\angle$ C. Show that AC = BD.

7. If a st. line bisects a line-segment at rt. $\angle$s, any point on the first line is equally distant from the ends of the line-segment.

32. **Theorem:—If one side of a triangle be produced, the exterior angle is greater than either of the interior and opposite angles.**

Let ABC be a $\triangle$, having BC produced to D.



It is required to show that the exterior angle ACD is greater than either of the interior and opposite $\angle$s ABC or BAC.

Bisect AC at E. Join BE, and produce BE to F, making $EF = BE$. Join FC.

In the two $\triangle$s AEB, CEF,

$$AE = EC,\ BE = EF, \text{ and } \angle AEB = \angle CEF,$$

$$\therefore \angle ECF = \angle EAB. \qquad (\S\ 30.)$$

$$\text{But } \angle ACD > \angle ECF,$$

$$\therefore \text{ also } \angle ACD > \angle BAC.$$

In the same manner, if AC be produced to G, it may be shown that $\angle BCG > \angle ABC$.

$$\text{But } \angle ACD = \angle BCG,$$

$$\therefore \angle ACD > \angle ABC.$$

33.—Exercises.

1. **Show that any two $\angle$s of a $\triangle$ are together less than two rt. $\angle$s.**

2. Can a $\triangle$ have two obtuse $\angle$s? Can a $\triangle$ have two rt. $\angle$s? Can a $\triangle$ have a rt. $\angle$ and also an obtuse $\angle$?

Show that a $\triangle$ must have at least two acute $\angle$s.

3. In an acute-∠d △ show that the ⊥ from a vertex to the opposite side cannot fall outside of the △.

4. In an obtuse-∠d △ show that the ⊥ from the vertex of the obtuse ∠ on the opposite side falls within the △, but that the ⊥ from the vertex of either acute ∠ on the opposite side falls outside of the △.

5. In a rt.-∠d △ where do the ⊥s from the vertices on the opposite sides fall?

6. Only one ⊥ can be drawn from a given point to a given st. line.

7. Not more than two line-segments each equal to the same given line-segment can be drawn from a given point to a given st. line.

8. D is a point taken within the △ ABC. Join DB, DC; and show, by producing BD to meet AC, that ∠ BDC > ∠ BAC.

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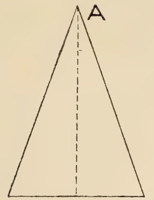
34. **Theorem :—The angles at the base of an isosceles triangle are equal to each other.**

Let ABC be an isosceles △ having AB = AC



It is required to show ∠ B = ∠ C.

Bisect ∠ A and produce the bisector to meet BC at D.

In the two △s ADB, ADC, AB = AC, AD is common, and ∠ BAD = ∠ CAD; ∴ ∠ B = ∠ C. (§ 30.)

35. It follows from the proof in § 34 that the bisector of the vertical ∠ of an isosceles △ bisects the base of the △ and cuts the base at rt. ∠s. It also bisects the area of the △.

If an isosceles $\triangle$ be folded along the bisector of the vertical $\angle$ as crease, the parts on one side of the crease will exactly fit the corresponding parts on the other side, and consequently **the bisector of the vertical $\angle$ of an isosceles $\triangle$ is an axis of symmetry of the $\triangle$.**

36.—Exercises.

1. An equilateral $\triangle$ is equiangular.

2. ABC is an equiangular $\triangle$, and points D, E, F, are taken in BC, CA, AB respectively, such that $BD = CE = AF$. Show that DEF is an equilateral $\triangle$.

3. Show that the exterior $\angle$s at the base of an isosceles $\triangle$ are equal to each other.

4. The opposite $\angle$s of a rhombus are equal to each other.

5. The $\triangle$ formed by joining the middle points of the sides of an isosceles $\triangle$ is isosceles.

6. Draw a figure having just two axes of symmetry. Draw a figure having just three axes of symmetry. Draw a figure having just four axes of symmetry. Draw a figure having many axes of symmetry.

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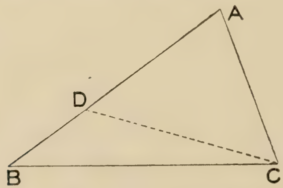
37. **Theorem :—If one side of a triangle be greater than another side, the angle opposite the greater side is greater than the angle opposite the less side.**

Let ABC be a $\triangle$ having $AB > AC$.



It is required to show that $\angle ACB > \angle ABC$.

From AB cut off $AD = AC$. Join DC.

In $\triangle$ ADC, $\because$ AD = AC,

$\therefore$ $\angle$ ADC = $\angle$ ACD. (§ 34.)

But $\angle$ ACB > $\angle$ ACD,

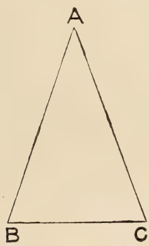
$\therefore$ $\angle$ ACB > $\angle$ ADC.

In $\triangle$ BDC $\because$ side BD is produced to A, $\therefore$ the exterior $\angle$ ADC > the interior and opposite $\angle$ BDC. (§ 32.)

But $\angle$ ACB > $\angle$ ADC; much more $\therefore$ is $\angle$ ACB > $\angle$ ABC.

38. **Theorem:—If two angles of a triangle be equal to each other, the sides opposite these equal angles are equal to each other.**

Let ABC be a triangle, having $\angle$ B = $\angle$ C.



It is required to show that side AC = side AB.

If AB be not = AC, let AB be the greater.

Then, by § 37, $\angle$ C > $\angle$ B; but this is not true, because it is contrary to the hypothesis.

$\therefore$ AB = AC.

39. Compare the propositions in §§ 34 and 38.

In § 34, the hypothesis is side AB = side AC, and the conclusion is $\angle$ B = $\angle$ C.

In § 38, the hypothesis is $\angle$ B = $\angle$ C, and the conclusion is side AB = side AC.

Thus in these propositions the hypothesis of each is the conclusion of the other.

When two propositions are such that the hypothesis of each is the conclusion of the other, they are said to be **converse propositions**; or each is said to be the converse of the other.

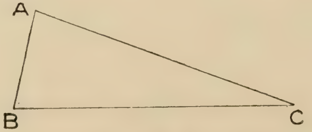
40. The converse of a true proposition may, or may not, be true. The converse propositions in §§ 34 and 38 are both true; but consider the true proposition:—All rt. ∠s are equal to each other; and its converse:—All equal ∠s are rt. ∠s. The last is easily seen to be untrue. Consequently proof must in general be given for each of a pair of converse propositions.

41. Consider the method of proof used in § 38. To prove that AC = AB we began by assuming that AC is not = AB and then showed that something absurd or contrary to the hypothesis must follow, and concluded that AC must be equal to AB.

This method in which we begin by assuming that the conclusion of the proposition is not true is called **the indirect method of demonstration.**

42. **Theorem:—If one angle of a triangle be greater than another angle of the same triangle, the side opposite the greater angle is greater than the side opposite the less.** (Converse of Theorem in § 37).

Let ABC be a △ having ∠ B > ∠ C.



It is required to show that AC > AC.

If AC be not > AB, then either AC = AB or AC < AB.

If AC = AB, then ∠ B = ∠ C (§ 34); but this is not true, because it is contrary to the hypothesis.

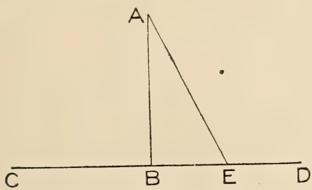
If AC < AB, then ∠ B < ∠ C (§ 37); but this is not true, because it is contrary to the hypothesis.

Then, because AC is neither equal to nor less than AB,

$$AC > AB.$$

43. **Theorem :—The perpendicular is the shortest distance from a given point to a given straight line.**

Let AB be the ⊥ from the point A to the st. line CD, and AE any other st. line from A terminated in CD.



It is required to show that AB < AE.

Because ABC is an exterior ∠ of the △ ABE, ∠ ABC > ∠ AEB. (§ 32.)

But ∠ ABC = ∠ ABE,

∴ ∠ ABE > ∠ AEB;

and consequently, by § 42,

$$AB < AE.$$

44.—Exercises.

1. An equiangular △ is equilateral.

2. ABCD is a quadrilateral, of which AD is the longest side, and BC the shortest. Show that ∠B > ∠D, and that ∠C > ∠ A.

3. The hypotenuse of a rt.-∠d △ is greater than either of the other two sides.

4. A st. line drawn from the vertex of an isosceles △ to any point in the base is less than either of the equal sides.

5. A st. line drawn from the vertex of an isosceles △ to any point in the base produced is greater than either of the equal sides.

6. If one side of a △ be less than another, the ∠ opposite the less side is acute.

7. If D be any point in the side, BC, of a △ ABC, the greater of the sides AB, AC, is greater than AD.

8. AB is drawn from A ⊥ CD. E, F are two points in CD on the same side of B, and such that BE $<$ BF. Show that AE $<$ AF. Prove the same proposition when E, F are on opposite sides of B.

9. Show that, if the ∠s ABC, ACB, at the base of the isosceles △ ABC be bisected by BD, CD, DBC is an isosceles △.

10. ABC is a △ having AB, AC produced to D, E respectively. The exterior ∠s DBC, ECB are bisected by BF, CF, which meet at F. Show that, if FB = FC, the △ ABC is isosceles.

11. ABC is a △ having AB $>$ AC. The bisector of ∠ A meets BC at D. Show that BD $>$ DC. Give a general statement of this proposition.

12. ABC is a △ having AB $>$ AC. If the bisectors of ∠s B, C meet at D, show that BD $>$ DC.

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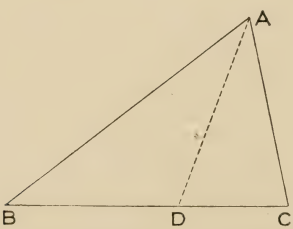
45. Theorem:—Any two sides of a triangle are together greater than the third side.

ABC is a △



It is required to show that AB + AC $>$ BC.

Bisect ∠ A and let the bisector meet BC at D.

ADC is an exterior ∠ of △ ABD, ∴ ∠ ADC > ∠ BAD.

But ∠ BAD = ∠ DAC.

∴ ∠ ADC > ∠ DAC, and consequently, by § 42, AC > DC.

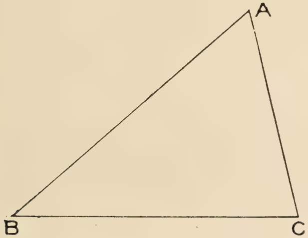
In the same manner it may be shown that AB > BD.

$$\therefore AB + AC > BD + DC; \text{ that is,}$$
$$AB + AC > BC.$$

In the same manner the pupil may show that AB + BC > AC and that AC + CB > AB.

46. **Theorem:—The difference between any two sides of a triangle is less than the third side.**

ABC is a △.



It is required to show that AB − AC < BC.

$$AB < AC + BC. \qquad (\S\ 45.)$$

From each of these unequals take AC,

$$\text{and } AB - AC < BC.$$

In the same manner the pupil may show that AB − BC < AC and that BC − AC < AB.

47.—Exercises.

1. Show that the sum of any three sides of a 4-gon is greater than the fourth side.

2. The sum of the four sides of a 4-gon is greater than the sum of its diagonals.

3. The sum of the diagonals of a 4-gon is greater than the sum of either pair of opposite sides.

4. The sum of the st. lines joining any point, except the intersection of the diagonals, to the four vertices of a 4-gon, is greater than the sum of the diagonals.

5. If any point within a △ be joined to the ends of a side of the △, the sum of the joining lines is less than the sum of the other two sides of the △.

6. If any point within a △ be joined to the three vertices of the △, the sum of the three joining lines is less than the perimeter of the △, but greater than half the perimeter.

7. The sum of any two sides of a △ is greater than twice the median drawn to the third side.

8. The median of a △ divides the vertical ∠ into parts, of which the greater is adjacent to the less side.

9. The perimeter of a △ is greater than the sum of the three medians.

10. A and B are two fixed points, and CD is a fixed st. line. Find the point P in CD, such that PA + PB is the least possible;

(*a*) When A and B are on opposite sides of CD;

(*b*) When A and B are on the same side of CD.

11. A and B are two fixed points, and CD is a fixed st. line. Find the point P in CD, such that the difference between PA and PB is the least possible;

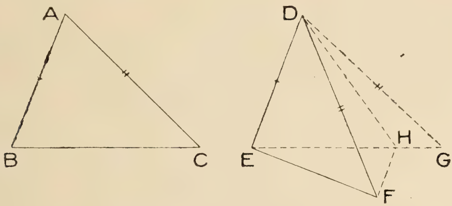
(*a*) When A and B are on the same side of CD;

(*b*) When A and B are on opposite sides of CD.

12. Prove the theorem in § 45 by producing BA to E, making AE = AC, and joining EC.

48. **Theorem:—If two triangles have two sides of one respectively equal to two sides of the other but the contained angle in one greater than the contained angle in the other, the base of the triangle which has the greater angle is greater than the base of the other.**

Let ABC, DEF be two $\triangle$s having AB = DE, AC = DF and $\angle$ BAC > $\angle$ EDF.



It is required to show that BC > EF.

Make $\angle$ EDG = $\angle$ BAC and cut off DG = AC, or DF. Join EG. Bisect $\angle$ FDG and let the bisector meet EG at H. Join FH.

In $\triangle$s ABC, DEG,

AB = DE, AC = DG and $\angle$ A = $\angle$ EDG.

$\therefore$ BC = EG.

In $\triangle$s FDH, GDH,

DF = DG, DH is common and $\angle$ FDH = $\angle$ GDH;

$\therefore$ FH = HG.

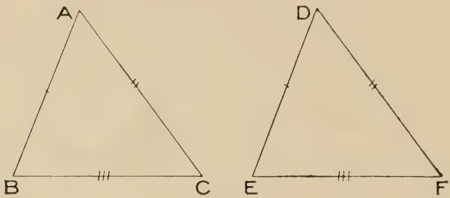
In $\triangle$ EHF, EH + HF > EF. (§ 45.)

But HF = HG; $\therefore$ EH + HG > EF, that is EG > EF. And BC has been shewn to be equal to EG, $\therefore$ BC > EF.

Second Case of the Equality of Triangles.

49. **Theorem :—If two triangles have the three sides of one respectively equal to the three sides of the other, the two triangles are congruent.**

Let ABC, DEF be two $\triangle$s having AB = DE, AC = DF and BC = EF.



It is required to prove $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E, $\angle$ C = $\angle$ F and area of $\triangle$ ABC = area of $\triangle$ DEF.

If $\angle$ A be not = $\angle$ D, let $\angle$ A > $\angle$ D.

Thus in $\triangle$s ABC, DEF :

AB = DE, AC = DF and $\angle$ A > $\angle$ D,

$\therefore$ BC > EF. (§ 48.)

But this is not true because it is contrary to the hypothesis;

$\therefore$ $\angle$ A = $\angle$ D.

Then, in $\triangle$s ABC, DEF,

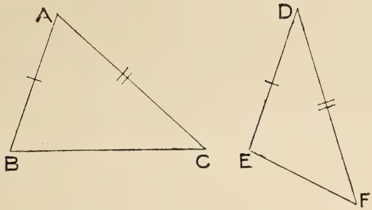
AB = DE, AC = DF and $\angle$ A = $\angle$ D,

$\therefore$, by § 30, the $\triangle$s are congruent.

---

50. **Theorem :—If two triangles have two sides of one respectively equal to two sides of the other but the base of one greater than the base of the other, the triangle which has the greater base has the greater vertical angle.** (Converse of Theorem in § 48.)

Let ABC, DEF be two $\triangle$s having AB = DE, AC = DF and BC > EF.



It is required to prove $\angle$ A > $\angle$ D.

If $\angle$ A be not > $\angle$ D, then either $\angle$ A = $\angle$ D, or $\angle$ A < $\angle$ D.

If $\angle$ A = $\angle$ D, then in the two $\triangle$s, AB = DE, AC = DF and $\angle$ A = $\angle$ D, and $\therefore$, by § 30, BC = EF.

But this is not true because it is contrary to the hypothesis.

$\therefore$ $\angle$ A is not equal to $\angle$ D.

If $\angle$ A < $\angle$ D, then in the two $\triangle$s, AB = DE, AC = DF and $\angle$ A < $\angle$ D, and $\therefore$, by § 48, BC < EF.

But this is not true because it is contrary to the hypothesis.

$\therefore$ $\angle$ A is not less than $\angle$ D.

Then since $\angle$ A is neither equal to nor less than $\angle$ D, $\angle$ A > $\angle$ D.

51.—Exercises.

1. If the opposite sides of a quadrilateral be equal, the opposite $\angle$s are equal.

2. A diagonal of a rhombus bisects each of the $\angle$s through which it passes.

3. If in a quadrilateral ABCD the sides AB, CD be equal and $\angle$ABC = $\angle$ BCD, $\angle$ CDA = $\angle$ DAB.

4. ABCD is a kite-shaped quadrilateral having AB = AD and BC = CD. Show that AC is an axis of symmetry in the figure.

5. On the same side of AB the two △s ACB, ADB have AC = BD, AD = BC and AD and BC meet at E. Show that AE = BE.

6. ABCD is a quadrilateral having AB = CD and ∠ BAD > ∠ ADC. Show that ∠ BCD > ∠ ABC.

7. In △ ABC, AB > AC and D is the middle point of BC. If any point P in the median AD be joined to B and C, BP > CP.

If AD be produced to any point Q show that BQ < QC.

8. D is a point in the side AB of the △ ABC. AC is produced to E making CE = BD. BE and CD are joined. Show that BE > CD.

9. Show that equal chords in a circle subtend equal ∠s at the centre.

10. If two chords of a circle be unequal the greater subtends the greater angle at the centre.

11. Two circles have a common centre at O. A, B are two points on the inner circumference and C, D two on the outer. ∠ AOC > ∠ BOD. Show that AC > BD.

12. Prove the theorem in § 49 by supposing △ DEF placed on △ ABC so that EF coincides with BC, and D is on the opposite side of the common base from A.

---

52. **Constructions.**—In the first five of the following constructions, use compasses and ruler only, and prove in each that your method is correct :—

1. **Bisect a given ∠.**

2. **Bisect a given line-segment.**

3. **Draw a ⊥ to a given st. line from a point in the line.**

4. **Draw a ⊥ to a given st. line from a point without the line.**

5. **At a given point in a given st. line make an ∠ = a given ∠.**

6. Make a △ congruent to a given △ ABC.

7. On a given base describe an isosceles △ such that the sum of the equal sides is equal to a given line-segment.

8. In the side BC of a △ ABC find a point E such that AE is half the sum of AB and AC.

9. Find a point in a given st. line such that its distances from two given points are equal to each other.

10. Find a point equally distant from three given points.

11. On a given base describe an isosceles △ having its altitude equal to a given line-segment.

12. On a given base construct a △ having one of the ∠s at the base equal to a given ∠, and the sum of the sides equal to a given line-segment.

13. On a given base construct a △ having one of the ∠s at the base equal to a given ∠ and the difference of the sides equal to a given line-segment.

CHAPTER II.

Theory of Parallels.

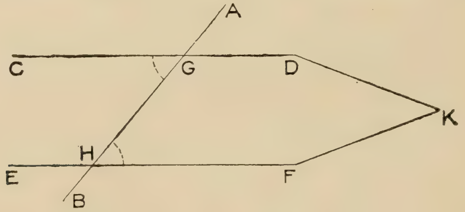
53. **Definition** :—Two straight lines in the same plane which do not meet when produced for any finite distance in either direction, are said to be **parallel** to each other.

54. **Definition** :—A straight line which cuts two, or more, other straight lines is called a **transversal.**

55. **Definition** :—A quadrilateral that has both pairs of opposite sides parallel to each other is called a **parallelogram.**

56. **Theorem** :—**If a transversal cut two straight lines making the alternate angles equal to each other, the two straight lines are parallel.**

Let the transversal AB cut the two st. lines CD, EF at G and H, making the ∠ CGH = the alternate ∠ GHF.



It is required to prove CD || EF.

If CD, EF be not ||, they will meet when produced either towards C, E or towards D, F.

If possible let them meet when produced towards D, F at K.

Then GHK is a △, and the exterior ∠ CGH is greater than the interior and opposite ∠ GHF.

But this is not true because it is contrary to the hypothesis.

∴ CD, EF do not meet when produced towards D, F.

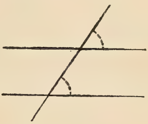
In the same manner it may be shown that they do not meet when produced towards C, E.

∴ CD ∥ EF.

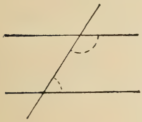
57.—Exercises.

1. Lines which are ⊥ to the same st. line are ∥ to each other.

2. **If a transversal cut two st. lines making an exterior ∠ equal to the interior and opposite ∠ on the same side of the transversal, the two lines are ∥.**



3. **If a transversal cut two st. lines making the two interior ∠s on the same side of the transversal supplementary, the two lines are ∥.**



4. If both pairs of opposite sides of a 4-gon are equal to each other, the 4-gon is a ∥gm.

5. A rhombus is a ∥gm.

6. If the diagonals of a 4-gon bisect each other, the 4-gon is a ∥gm.

7. A square is a ∥gm.

8. No two st. lines drawn from two vertices of a △, and terminated in the opposite sides, can bisect each other.

9. Using compasses and ruler only, draw from a given point a st. line ∥ to a given st. line. Show that your method is correct.

10. Using the set-square, draw from a given point a st. line ∥ to a given st. line. Show that your method is correct.

---

58. The following is a statement of a fundamental property, or axiom, of parallel straight lines :—

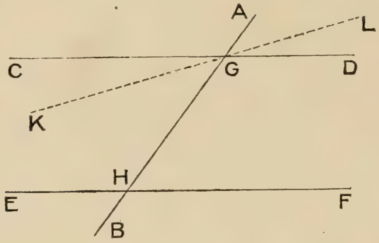
Through any point one and only one straight line can be drawn parallel to a given straight line.

And consequently :—

No two intersecting straight lines can be parallel to the same straight line.

59. **Theorem :—If a transversal cut two parallel straight lines, the alternate angles are equal to each other.**

Let the transversal AB cut the two || st. lines CD, EF at G, H respectively.



It is required to prove ∠ CGH = ∠ GHF.

If ∠ CGH be not equal to ∠ GHF, make the ∠ KGH = ∠ GHF, and produce KG to L.

Then because AB cuts KL and EF, making ∠ KGH = the alternate ∠ GHF.

KL is || to EF. (§ 56.)

But CD is, by hypothesis, || to EF.

That is, two intersecting st. lines, KL and CD, are both || EF, which is contrary to the axiom of § 58.

∴ ∠ CGH = ∠ GHF.

60.—Exercises.

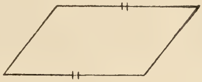
1. **If a transversal cut two ∥ st. lines, it makes an exterior ∠ = the interior and opposite ∠ on the same side of the transversal.**

2. **If a transversal cut two ∥ st. lines, it makes the sum of the two interior ∠s on the same side of the transversal equal to two rt. ∠s.**

3. If a st. line be ⊥ to one of two ∥ st. lines, it is also ⊥ to the other.

4. **St. lines which are ∥ to the same st. line are ∥ to each other.**

5. **St. lines which join the ends of two equal and ∥ line-segments towards the same parts are themselves equal and ∥.**



6. If the bisector of an exterior ∠ of a △ be ∥ to the opposite side, the △ is isosceles.

7. Through a point on the bisector of an ∠ a line is drawn ∥ to one of the arms. Prove that the △ thus formed is isosceles.

8. Any st. line ∥ to the base of an isosceles △ makes equal ∠s with the sides, or the sides produced.

9. Find a point B in a given st. line CD such that, if AB be drawn to B from a given point A, the ∠ ABC will be equal to a given ∠.

10. Construct a △ having two of its ∠s respectively equal to two given ∠s, and the length of the ⊥ from the vertex of the third ∠ to the opposite side equal to a given line-segment.

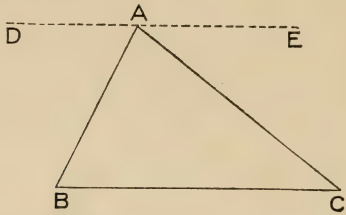
11. Construct a rt.-∠d △, having given one side and the opposite ∠.

12. If one ∠ of a ∥gm be a rt. ∠, the other three ∠s are also rt. ∠s.

THE SUM OF THE ANGLES OF A TRIANGLE.

61. **Theorem :—The three angles of a triangle are together equal to two right angles.**

Let ABC be a $\triangle$.



It is required to show that $\angle$ A + $\angle$ B + $\angle$ C = two rt. $\angle$ s.

Through A draw DE $\parallel$ BC.

AB cuts the $\parallel$ lines DA, BC,

$\therefore$ $\angle$ B = alternate $\angle$ DAB.

AC cuts the $\parallel$ lines AE, BC,

$\therefore$ $\angle$ C = alternate $\angle$ CAE.

$\therefore$ $\angle$ B + $\angle$ C + $\angle$ BAC = $\angle$ DAB + $\angle$ BAC + $\angle$ CAE

= two right $\angle$ s.

62.—Exercises.

1. **If one side of a $\triangle$ be produced, the exterior $\angle$ is equal to the sum of the two interior and opposite $\angle$s.**

2. **If two $\triangle$s have two $\angle$s of one respectively equal to two $\angle$s of the other, the third $\angle$ of one is equal to the third $\angle$ of the other.**

3. **The sum of the $\angle$s of a 4-gon is equal to four rt. $\angle$s.**

4. **The sum of the $\angle$s of a polygon of n sides is $2n-4$ rt. $\angle$s.** (Note: join each vertex to a point within the polygon).

5. Each $\angle$ of an equilateral $\triangle$ is an $\angle$ of 60°.

6. Find the number of degrees in each ∠ of: (*a*) an equiangular pentagon; (*b*) an equiangular hexagon; (*c*) an equiangular octagon; (*d*) an equiangular decagon.

7. Each ∠ of an equiangular polygon contains 162°. Find the number of sides.

8. Each ∠ of an equiangular polygon contains 170° Find the number of sides.

9. A st. line drawn ⊥ to BC, the base of an isosceles △ ABC, cuts AB at X and CA produced at Y. Show that AXY is an isosceles △.

10. ACB is a rt.-∠d △ having the rt. ∠ at C. Through X, the middle point of AC, XY is drawn ∥ CB cutting AB at Y. Show that Y is the middle point of AB.

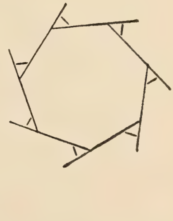
11. The middle point of the hypotenuse of a rt.-∠d △ is equidistant from the three vertices.

12. The st. line joining the middle points of two sides of a △ is ∥ to the third side.

13. Construct a rt.-∠ △, having the hypotenuse equal to one given line-segment, and the sum of the other two sides equal to another given line-segment.

14. If one ∠ of a △ equals the sum of the other two, show that the △ is a rt.-∠d △.

15. Show that the bisectors of the two acute ∠s of a rt.-∠d △ contain an ∠ of 135°.



16. If the sides of a polygon be produced in order the same way round, the sum of the exterior ∠s thus formed is equal to four rt. ∠s.

17. If the opposite angles of a 4-gon are equal the 4-gon is a ∥gm.

18. From C, the middle point of a line-segment AB, CD is drawn in any direction and equal to CA or CB. Show that ADB is a rt. ∠.

19. On AB, AC, sides of a △ ABC, equilateral △s ABD, ACE are described externally. Show that DC = BE.

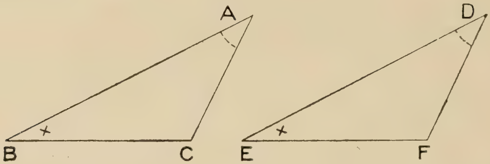
20. AB is any chord of a circle of which the centre is O. AB is produced to C so that BC = BO. CO is joined, cutting the circle at D and is produced to cut it again at E. Show that ∠ AOE = three times ∠ BCD.

21. If the exterior ∠s at B and C of a △ ABC be bisected and the bisectors be produced to meet at D, the ∠ BDC equals half the sum of ∠s ABC, ACB.

THIRD CASE OF THE EQUALITY OF TRIANGLES.

63. **Theorem:—If two triangles have two angles and a side of one respectively equal to two angles and the corresponding side of the other, the triangles are congruent.**

Let ABC, DEF be two △s having ∠ A = ∠ D, ∠ B = ∠ E, and BC = EF.



It is required to prove ∠ C = ∠ F, AB = DE, = AC DF, and area of △ ABC = area of △ DEF.

∵ ∠ A = ∠ D and ∠ B = ∠ E; ∴, by Ex. 2, § 62, ∠ C = ∠ F.

Apply △ ABC to △ DEF so that vertex B falls on vertex E, and BC falls along EF.

Then, ∵ BC = EF, vertex C must fall on vertex F.

∵ ∠ B = ∠ E, BA must fall along ED, and A must be somewhere on line ED.

∵ ∠ C = ∠ F, CA must fall along FD, and A must be somewhere on line FD.

But D is the only point common to ED and FD, and consequently A must fall on D.

$\therefore$ $\triangle$ ABC coincides with $\triangle$ DEF, and the two $\triangle$s are congruent.

64.—Exercises.

1. If the bisector of an $\angle$ of a $\triangle$ be $\perp$ to the opposite side, the $\triangle$ is isosceles.

2. Any point in the bisector of an $\angle$ is equidistant from the arms of the $\angle$.

3. In the base of a $\triangle$ find a point that is equidistant from the two sides.

4. In a given st. line find a point that is equidistant from two other given st. lines.

5. Within a $\triangle$ find a point that is equally distant from the three sides of the $\triangle$.

6. Without a $\triangle$ find three points each of which is equally distant from the three st. lines that form the $\triangle$.

7. **The opposite sides of a $\parallel$gm. are equal to each other.**

8. **The opposite $\angle$s of a $\parallel$gm. are equal to each other.**

9. **The diagonal of a $\parallel$gm. bisects the area of the $\parallel$gm.**

10. **The diagonals of a $\parallel$gm. bisect each other.**

11. **The st. line joining the middle points of two sides of a $\triangle$ equals half the third side.** (See Ex. 12, § 62.)

12. The ends of the base of an isosceles $\triangle$ are equidistant from the opposite sides.

13. If two sides of a 4-gon be $\parallel$, and the other two be equal to each other but not $\parallel$, the diagonals of the 4-gon are equal.

14. Through a given point draw a st. line, such that the part of it intercepted between two given $\parallel$ st. lines is equal to a given line-segment.

Show that, in general, two such lines can be drawn.

15. Through a given point draw a st. line that shall be equidistant from two other given points.

Show that, in general, two such lines can be drawn.

16. Draw a st. line || to a given st. line, and such that the part of it intercepted between two given intersecting lines is equal to a given line-segment.

17. BAC is a given ∠, and P is a given point. Draw a line-segment terminated in the st. lines AB, AC and bisected at P.

18. Construct a △ having given the middle points of the three sides.

The Ambiguous Case in the Comparison of Triangles.

65. **Theorem :—If two triangles have two sides of one respectively equal to two sides of the other and have the angles opposite one pair of equal sides equal to each other, the angles opposite the other pair of equal sides are either equal or supplementary.**

Let ABC, DEF be two △s having AB = DE, AC = DF and ∠ B = ∠ E.

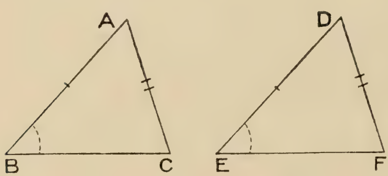


FIG. 1.

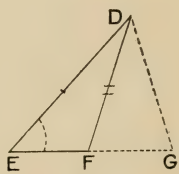


FIG. 2.

It is required to prove that either ∠ C = ∠ F, or ∠ C + ∠ F = two rt. ∠s.

Case I.—Suppose ∠ A = ∠ D. (Fig. 1.)

Then in the two △s ABC, DEF,

∠ A = ∠ D and ∠ B = ∠ E,

∴ ∠ C = ∠ F. (Ex. 2, § 62.)

Case II. Suppose $\angle A$ not $= \angle D$. (Fig. 2.)

Make $\angle EDG = \angle BAC$, and produce its arm to meet EF, produced if necessary at G.

In the $\triangle$s ABC, DEG,

$\angle A = \angle EDG$, $\angle B = \angle E$, and $AB = DE$,

$\therefore \angle C = \angle G$ and $AC = DG$. (§ 63.)

But $DF = AC$ (hypothesis),

$\therefore DF = DG$.

Then, $\because$ DFG is an isosceles $\triangle$,

$\angle G = \angle DFG$.

But it has been shown that $\angle C = \angle G$,

$\therefore \angle C = \angle DFG$.

And $\angle DFG + \angle DFE =$ two rt. $\angle$s,

$\therefore \angle C + \angle DFE =$ two rt. $\angle$s.

66.—Exercises.

1. **If two rt.-$\angle$d $\triangle$s have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the $\triangle$s are congruent.**

2. If the bisector of the vertical $\angle$ of a $\triangle$ also bisects the base, the $\triangle$ is isosceles.

3. If two $\triangle$s have two sides of one respectively equal to two sides of the other and the $\angle$s opposite the greater pair of equal sides equal to each other, the $\triangle$s are congruent.

4. Construct a $\triangle$ having given two sides and the $\angle$ opposite one of them.

   When will there be: (*a*) no solution, (*b*) two solutions, (*c*) only one solution?

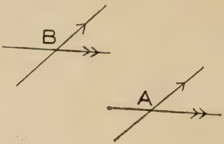
5. If two $\angle$s of a $\triangle$ be bisected and the bisectors be produced to meet, the line joining the point of intersection to the vertex of the third $\angle$ bisects that third $\angle$.

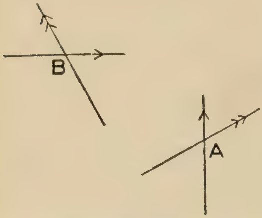
6. If two exterior $\angle$s of a $\triangle$ be bisected and the bisectors be produced to meet, the line joining the point of intersection of the bisectors to the vertex of the third $\angle$ of the $\triangle$ bisects that third $\angle$.

67.—Exercises.

1. If a line-segment be terminated by two ||s, all line-segments drawn through its middle point and terminated by the same ||s are bisected at that point.

2. **If two lines intersecting at A be respectively || to two lines intersecting at B, each ∠ at A is either equal to or supplementary to each ∠ at B.**





3. **If two lines intersecting at A be respectively ⊥ to two lines intersecting at B, each ∠ at A is either equal to or supplementary to each ∠ at B.**

4. If from any point in the bisector of an ∠ line-segments be drawn || to the arms of the ∠ and terminated by the arms, these line-segments are equal to each other.

5. In the base of a △ find a point such that the line-segments drawn from that point || to the sides of the △ and terminated by the sides are equal to each other.

6. One ∠ of an isosceles △ is half each of the others. Calculate the ∠s.

7. If the ⊥ from the vertex of a △ to the base falls within the △, the segment of the base adjacent to the greater side of the △ is the greater.

8. If a star-shaped figure be formed by producing the alternate sides of a polygon of n sides, the sum of the ∠s at the points of the star is $2n - 8$ rt. ∠s.

9. **The diagonals of a rectangle are equal to each other.**

10. The bisectors of the ∠s of a ||gm. form a rectangle, the diagonals of which are || to the sides of the original ||gm.

11. From A, B the ends of a line-segment ⊥s AC, BD are drawn to any st. line. E is the middle point of AB. Show that EC = ED.

12. If through a point within a △ three line-segments be drawn from the vertices to the opposite sides, the sum of these line-segments is greater than half the perimeter of the △.

13. A, D are the centres of two circles, and AB, DE are two || radii. EB cuts the circumferences again at C, F. Show that AC || DF.

14. The bisectors of the interior ∠s of a 4-gon form a 4-gon of which the opposite ∠s are supplementary.

15. In a given square inscribe an equilateral △ having one vertex at a vertex of the square.

16. Through two given points draw two st. lines, forming an equilateral △ with a given st. line.

17. Draw an isosceles △ having its base in a given st. line, its altitude equal to a given line-segment, and its equal sides passing through two given points.

18. If a ⊥ be drawn from one end of the base of an isosceles △ to the opposite side, the ∠ between the ⊥ and the base = half the vertical ∠ of the △.

19. If any point P in AD the bisector of the ∠ A of △ ABC be joined to B and C, the difference between PB and PC is less than the difference between AB and AC.

20. If any point P in the bisector of the exterior ∠ at A in the △ ABC be joined to B and C, PB + PC > AB + AC.

21. BAC is a rt. ∠ and D is any point. DE is drawn ⊥ AB and produced to F, making EF = DE. DG is drawn ⊥ AC and produced to H, making GH = DG. Show that F, A, H are in the same st. line.

22. Construct a △ having its perimeter equal to a given line-segment, and its ∠s respectively equal to the ∠s of a given △.

23. If the diagonals of a ||gm are equal to each other, the ||gm is a rectangle.

24. If the arms of one ∠ be respectively || to the arms of another ∠, the bisectors of the ∠s are either || or ⊥.

25. In a given △ inscribe a ||gm the diagonals of which intersect at a given point.

26. Show that the $\perp$s from the centre of a circle to two equal chords are equal to each other.

27. Construct a 4-gon having its sides equal to four given line-segments and one $\angle$ equal to a given $\angle$.

28. The bisector of $\angle$ A of $\triangle$ ABC meets BC at D and BC is produced to E. Show that $\angle$ ABC + $\angle$ ACE = twice $\angle$ ADC.

29. The bisectors of $\angle$s A and B of $\triangle$ ABC intersect at D. Show that $\angle$ ADB = 90° + half of $\angle$ C.

30. The sides AB, AC of a $\triangle$ ABC are bisected at D, E; and BE, CD are produced to F, G, so that EF = BE and DG = CD. Show that F, A, G are in the same st. line, and that FA = AG.

31. ABC is an isosceles $\triangle$. AE, AD are equal parts cut off from AB, AC respectively, BD, CE cut at F. Show that FBC and FDE are isosceles $\triangle$s.

32. In a $\triangle$ ABC, the bisector of $\angle$ A and the line bisecting BC at rt. $\angle$s meet at D. DE, DF are drawn $\perp$ AB, AC respectively. Show that AE = AF and that BE = CF.

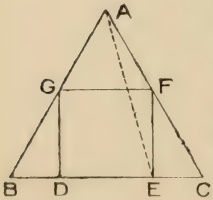
33. ABC is a $\triangle$ having AB > AC. AB, AC are produced to D, E respectively so that BD = CE. Show that DC > BE.

34. Through a given point draw a st. line cutting two intersecting st. lines and forming an isosceles $\triangle$ with them.

Show that two such lines can be drawn through the given point.

35. If ACB be a st. line and ACD, BCD two adjacent $\angle$s, any $\parallel$ to AB will meet the bisectors of these $\angle$s in points equally distant from where it meets CD.

36. Inscribe a square in a given equilateral $\triangle$.



NOTE.—Draw a sketch as in the diagram given here. Join AE.

What is the number of degrees in $\angle$ CAE?

37. ABC is a $\triangle$, AX is $\perp$ BC and AD bisects $\angle$ BAC. Show that $\angle$ XAD equals half the difference of $\angle$s B and C.

38. Construct a $\parallel$gm. having its diagonals and a side respectively equal to three given line-segments.

39. Find a point in each of two || st. lines such that the two points are equally distant from a given point and the line-segment joining them subtends a rt. ∠ at the given point.

40. P, Q are two given points on the same side of a given st. line BC. Find the position of a point A in BC such that ∠ PAB = ∠ QAC.

NOTE.—If P, Q are two points on a billiard table and BC the side of the table, a ball starting from P and reflected from BC at A would pass through Q.

41. Find the path of a billiard ball which, starting from a given point, is reflected from the four sides of the table and passes through another given point.

42. BAC is a given ∠ and C, D are two given line-segments. Find a point P such that its distances from AB, AC equal C, D respectively.

CHAPTER III.

Areas of Parallelograms and Triangles.

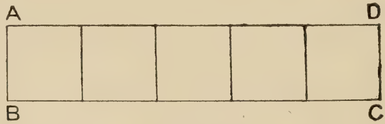
68. A square unit of area is a square, each side of which is equal to a unit of length.

Examples :—A square inch is a square each side of which is one inch ; a square centimetre is a square each side of which is one centimetre.

The acre is an exceptional case.

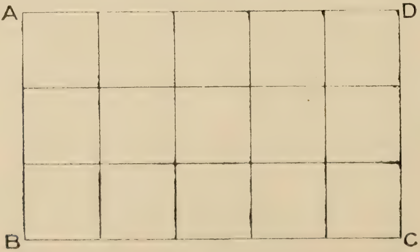
69. A numerical measure of any area is the number of times the area contains some unit of area.

ABCD is a rectangle one centimetre wide and five centimetres long.



This rectangle is a strip divided into five square centimetres, and consequently the numerical measure of its area in square centimetres is 5.

70. ABCD is a rectangle 3 cm. wide and 5 cm. long.



This rectangle is divided into 5 strips of 3 sq. cm. each, or into 3 strips of 5 sq. cm. each, and consequently the measure of the area in square centimetres is 5×3, or 3×5.

Similarly, if the length of a rectangle is 2.34 inches and its breadth .56 of an inch, the one-hundreth of an inch may be taken as the unit and the rectangle can be divided into 234 strips each containing 56 square one-hundreths of an inch. The measure of the area then is 234×56 of these small squares, ten thousand (100×100) of which make one square inch.

This method of expressing the area of a rectangle may be carried to any degree of approximation, so that in all cases the numerical measure of its area is equal to the product of its length by its breadth.

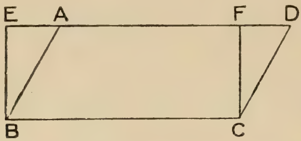
71. The finding of the areas of triangles, quadrilaterals, and polygons, can by geometrical constructions be made to depend in each case on finding the area of a rectangle.

From the theorems in §§ 72, 73, it will be seen how the areas of parallelograms and triangles depend for their calculation on equivalent rectangles.

In § 81 a triangle is constructed equal in area to any quadrilateral, and the same method developed in examples 5, 6, 7 and 8 of § 82 will give a triangle equivalent in area to any polygon, and thence the measure of the area may be found.

72. **Theorem :—The area of a parallelogram is equal to that of a rectangle on the same base and of the same altitude.**

Let ABCD be a ||gm and EBCF a rectangle on the same base BC and of the same altitude EB.



It is required to show that area of ||gm ABCD = area of rect. EBCF.

In $\triangle$s ABE, DCF.

AB = DC (why?), $\angle$ EAB = $\angle$ FDC (why?), and $\angle$ BEA = $\angle$ CFD.

$\therefore$ $\triangle$ ABE = $\triangle$ DCF. (§ 63.)

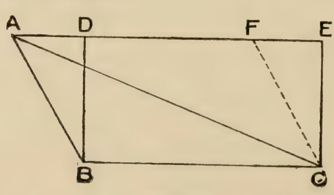
4-gon EBCD − $\triangle$ ABE = ||gm ABCD.

4-gon EBCD − $\triangle$ DCF = rect. EBCF;

and as equal parts have been taken from the same thing, the remainders must be equal;

$\therefore$ ||gm ABCD = rect. EBCF.

73. **Theorem :—The area of a triangle is half that of the rectangle on the same base and of the same altitude as the triangle.**



Let ABC be a $\triangle$ and DBCE a rectangle on the same base and of the same altitude BD.

It is required to show that area of $\triangle$ ABC equals half that of rectangle DBCE.

Through C draw CF $\parallel$ BA.

AC is a diagonal of $\parallel$gm. ABCF and $\therefore$ $\triangle$ ABC = half $\parallel$gm ABCF.

But $\parallel$gm ABCF = rect. DBCE. (§ 72.)

$\therefore$ $\triangle$ ABC = half of rect. DBCE.

74. If a be the measure of the base of a $\parallel$gm. and b the measure of its altitude, the measure of its area is ab.

If a be the measure of the base of a $\triangle$ and b the measure of its altitude, the measure of its area is $\frac{1}{2}ab$.

75. In a rectangle any side may be called the base, and then either of the adjacent sides is the altitude.

A rectangle as ABCD is commonly represented by the symbol AB.BC, where AB and BC may be taken to represent the number of units in the length and the breadth respectively.

Any side of a triangle may be taken as the base, the altitude being the perpendicular from the opposite vertex on the side taken as base.

76.—Exercises.

1. *(a)* **$\parallel$gms on the same base and having equal altitudes are equal in area.**

(b) **$\parallel$gms on equal bases in the same st. line and having equal altitudes are equal in area.**

2. *(a)* **$\triangle$s on the same base and having equal altitudes are equal in area.**

(b) **$\triangle$s on equal bases in the same st. line and having equal altitudes are equal in area.**

3. If a $\triangle$ and a $\parallel$gm are on the same base and between the same $\parallel$s, the area of the $\triangle$ is half that of the $\parallel$gm.

4. **A median bisects the area of the △.**

5. On the same base with a ||gm construct a rectangle equal in area to the ||gm.

6. On the same base with a given ||gm, construct a ||gm equal in area to the given ||gm, and having one of its sides equal to a given line-segment.

7. Construct a rect. equal in area to a given ||gm, and having one of its sides equal to a given line-segment.

8. On the same base as a given △ construct a rect. equal in area to the △.

9. Construct a rect. equal in area to a given △, and having one of its sides equal to a given line-segment.

10. On the same base with a ||gm construct a rhombus equal in area to the ||gm.

11. Construct a rhombus equal in area to a given ||gm, and having each of its sides equal to a given line-segment.

12. On the same base with a given △, construct a rt.-∠d △ equal in area to the given △.

13. On the same base with a given △, construct an isosceles △ equal in area to the given △.

14. If, in the ||gm. ABCD, P be any point between AB, CD produced indefinitely, the sum of the △s PAB, PCD equals half the ||gm; and if P be any point not between AB, CD, the difference of the △s PAB, PCD equals half the ||gm.

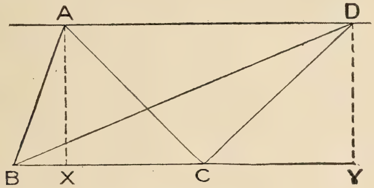
15. AB and ECD are two || st. lines; BF, DF are drawn || AD, AE respectively; prove that △s ABC, DEF are equal to each other.

16. On the same base, with a given △, construct a △ equal in area to the given △, and having its vertex in a given st. line.

17. If two △s have two sides of one respectively equal to two sides of the other and the contained ∠s supplementary, the △s are equal in area.

77. **Theorem:—If two equal triangles be on the same side of a common base, the straight line joining their vertices is parallel to the common base.**

Let ABC, DBC be two equal $\triangle$s on the same side of the common base BC.



It is required to prove AD $\parallel$ BC.

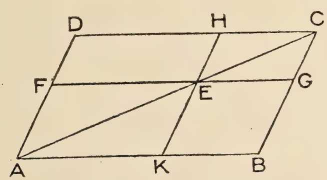
Draw AX and DY $\perp$ BC.

$$\triangle\ ABC = \tfrac{1}{2}\ \text{rect. } BC.AX. \qquad (\S\ 73.)$$
$$\triangle\ DBC = \tfrac{1}{2}\ \text{rect. } BC.DY;$$
$$\text{but } \triangle\ ABC = \triangle\ DBC,$$
$$\therefore \tfrac{1}{2}\ BC.AX = \tfrac{1}{2}\ BC.DY$$
$$\text{and hence } AX = DY,$$

that is, AX and DY are both = and $\parallel$ to each other

$$\therefore AD \parallel XY. \qquad (\S\ 60,\ \text{Ex. } 5.)$$

78. If, through any point E, in the diagonal AC of a parallelogram BD, two straight lines FEG, HEK be drawn parallel respectively to the sides DC, DA of the parallelogram,



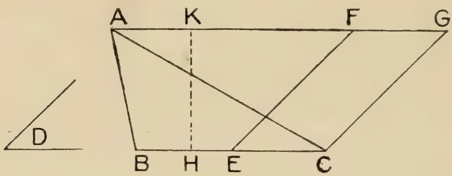
the $\parallel$gms FK and HG are said to be parallelograms about the diagonal AC, and the $\parallel$gms DE, EB are called the complements of the $\parallel$gms FK, HG, which are about the diagonal.

79. **Theorem:—The complements of the parallelograms about the diagonal of any parallelogram are equal to each other.**

[*The proof of this theorem is left for the pupil.*]

80. **Problem:—To construct a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.**

Let ABC be the given $\triangle$ and D the given $\angle$.



It is required to construct a ||gm. equal in area to $\triangle$ ABC and having one $\angle$ equal to $\angle$ D.

Through A draw AFG || BC. Bisect BC at E. At E make $\angle$ CEF = $\angle$ D. Through C draw CG || EF.

Draw any line HK $\perp$ to the two || st. lines:

HK is the common altitude of the ||gm. FC and the $\triangle$ ABC.

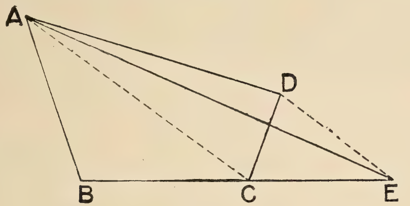
||gm. FC = rect. EC.HK. (§ 72.)

$= \frac{1}{2}$ rect. BC.HK, $\because$ EC $= \frac{1}{2}$ BC,

$= \triangle$ ABC. (§ 73)

81. **Problem :—To construct a triangle equal in area to a given quadrilateral.**

Let ABCD be the given 4-gon.



It is required to construct a △ equal in area to ABCD.

Join AC. Through D draw DE ∥ AC and meeting BC produced at E. Join AE.

△ ABE = 4-gon ABCD.

[*The proof is left for the pupil.*]

82.—Exercises.

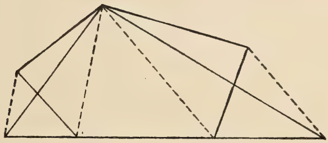
1. **If two equal △s be on equal segments of the same st. line and on the same side of the line, the st. line joining their vertices is ∥ to the line containing their bases.**

2. Through P, a point within the ∥gm ABCD, EPF is drawn ∥ AB, and GPH is drawn ∥ AD. If ∥gm AP = ∥gm PC, show that P is on the diagonal BD. (Converse of Theorem in § 79.)

3. Construct a rect. equal in area to a given △.

4. Construct a rect. equal in area to a given 4-gon.

5. Construct a △ equal in area to a given pentagon.



Thence make a rect. equal in area to the pentagon.

6. Construct a 4-gon equal in area to a given hexagon.

7. Construct a pentagon equal in area to a given heptagon.

8. **Show how a △ may be described equal in area to any given polygon; and thence show how to find the area of the polygon.**

9. Of equal △s on the same base the isosceles △ has the least perimeter.

10. On one side of a given △ construct a rhombus equal in area to the given △.

11. Bisect a given △ by a st. line drawn through a given point in one side of the △.

12. The diagonals of ||gms about the diagonal of a ||gm are ||.

13. ||gms about the diagonal of a square are squares.

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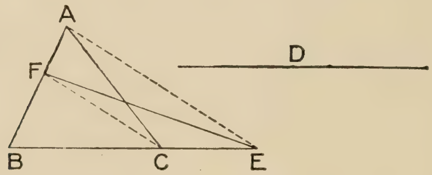
83. **Problem:—To construct a triangle equal in area to a given triangle and having one of its sides equal to a given line-segment.**

Let ABC be the given △ and D the given line-segment,



It is required to make a △ = △ ABC and having one side = D.

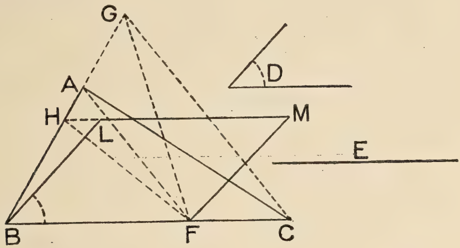
From BC, produced if necessary, cut off BE = D. Join AE. Through C draw CF || EA and meeting BA, or BA produced at F. Join FE.

FBE is the required △.

[*The proof is left for the pupil.*]

84. **Problem:—On a line-segment of given length to make a parallelogram equal in area to a given triangle and having an angle equal to a given angle.**

Let ABC be the given △, E the given line-segment and D the given ∠.



It is required to make a ||gm. equal in area to △ ABC, having one side equal in length to E and one ∠ equal to D.

From BC, produced if necessary, cut off BF = E. Join AF. Through C draw CG || FA meeting BA, or BA produced, at G. Join GF, Bisect BG at H. Through H draw HM || BC. At B make ∠ CBL = ∠ D. Through F draw FM || BL

LBFM is the required ||gm.

Join HF.

△s GAF, AFC are on the same base AF and have the same altitude, ∴ they are equal. (§ 76, Ex. 2.)

To each of these equal △s add the △ ABF, and △ GBF = △ ABC.

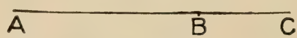
△ GBF = twice △ HBF, (§ 76, Ex. 4)
= ||gm LBFM, (§ 76, Ex. 3)
∴ ||gm LBFM = △ ABC.

Also ∠ LBF = ∠ D and side BF = E.

85. A rectangle is said to be contained by two line-segments when its length is equal to one of the line-segments, and its breadth is equal to the other.

86. **Theorem :—The square on the sum of two line-segments equals the sum of the squares on the two line-segments increased by twice the rectangle contained by the line-segments.**

Let AB, BC be the two line-segments placed in the same st. line so that AC is their sum.



It is required to show that

$$AC^2 = AB^2 + BC^2 + 2 \, . \, AB \, . \, BC.$$

Let a, b represent the number of units of length in AB, BC respectively.

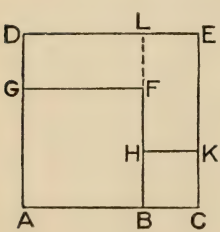
Area of the square on AC :

$= (a + b)^2$

$= a^2 + b^2 + 2\,ab$

= area of square on AB + area of square on BC + twice the area of the rectangle contained by AB and BC.

87. Illustration of Theorem in § 86.



AC = AB + BC.

ADEC is the square on the sum of AB and BC.

AGFB is the square on AB.

BHKC is the square on BC.

Produce BF to L.

GD = AD − AG = AC − AB = BC, and GF = AB,

∴ GL = rect. AB.BC.

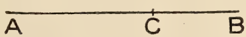
KE = CE − CK = AC − BC = AB, and HK = C,

∴ HE = rect. AB.BC.

Square on AC = AE = AF + BK + GL + HE = square on AB + square on BC + twice rect. AB.BC.

88. **Theorem :—The square on the difference of two line-segments equals the sum of the squares on the two line-segments diminished by twice the rectangle contained by the line-segments.**

Let AB, BC be the two line-segments of which AB is the greater, and let them be placed in the same st. line, and so that AC is their difference.



It is required to show that

$$AC^2 = AB^2 + BC^2 - 2.AB.BC.$$

Let a, b represent the number of units of length in AB, BC respectively.

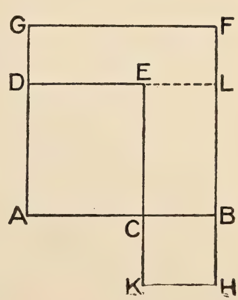
Area of square on AC $= (a - b)^2$,

$= a^2 + b^2 - 2ab$,

= area of square on AB + area of square on BC − twice the area of the rectangle contained by AB and BC.

89. Illustration of Theorem in § 88.

ADEC is the square on the difference of AB and BC.



AGFB is the square on AB.

BHKC is the square on BC

Produce DE to L.

DG = AG − AD = AB − AC = BC and DL = AB,

∴ DF = rect. AB.BC.

KE = KC + CE = BC + AC = AB, and KH = BC,

$\therefore$ KL = rect. AB.BC.

Square on AC = AE = AF + KB − (DF + KL) = square on AB + square on BC − twice the rect. AB.BC.

90. **Theorem:—The difference of the squares on two line-segments equals the rectangle of which the length is the sum of the line-segments and the breadth is the difference of the line-segments.**

Let A, B be two st. lines, of which A > B.

A ———————— B ————

It is required to show that the square on A − the square on B = the rect. contained by A + B and A − B.

Let a, b represent the number of units in A and B respectively.

The difference of the squares on A and B

$$= a^2 - b^2$$

$$= (a + b)(a - b)$$

= the area of the rectangle contained by A + B and A − B.

91.—Exercises.

1. Draw a diagram illustrative of the theorem in § 90.

2. The square on the sum of three line-segments equals the sum of the squares on the three line-segments increased by twice the sum of the rectangles contained by each pair of the line-segments.

Illustrate by diagram.

3. The sum of the squares on two unequal line-segments > twice the rectangle contained by the two line-segments.

4. The sum of the squares on three unequal line-segments > the sum of the rectangles contained by each pair of the line-segments.

5. Construct a rectangle equal to the difference of two given squares.

92.—Exercises.

1. If a 4-gon be bisected by each of its diagonals it is a ||gm.

2. If any point P in the diagonal AC of the ||gm ABCD be joined to B and D, the ||gm. is divided into two pairs of equal △s.

3. The diagonals of a ||gm. divide the ||gm. into four equal parts.

4. If two sides of a 4-gon are || to each other, the st. line joining their middle points bisects the area of the 4-gon.

5. If two sides of a 4-gon are || to each other, the st. line joining their middle points passes through the intersection of the diagonals.

6. If P is any point in the side AB of ||gm. ABCD, and PC, PD are joined,

$$\triangle\ PAD + \triangle\ PBC = \triangle\ PDC.$$

7. Prove that the following method of bisecting a 4-gon by a st. line drawn through one of its vertices is correct :—Let ABCD be the 4-gon. Join AC, BD. Bisect BD at E. Through E draw EF || AC and meeting BC, or CD at F. Join AF. AF bisects the 4-gon.

8. If the diagonals of ||gm ABCD cut at O, and P is any point within the △ AOB, △ CPD = △ APB + △ APC + △ BPD.

Note :—Join PO.

9. Any st. line drawn through the point of intersection of the diagonals of a ||gm bisects the ||gm.

10. Bisect a ||gm. by a st. line drawn from a given point.

11. Construct a △ having two sides and the median drawn to one of these sides equal to three given line-segments.

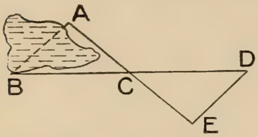
12. Construct a △ having two sides and the median drawn to the third side equal to three given line-segments.

13. ABCD is a 4-gon, having AB || CD. E is the middle point of BC. Show that △ AED = half the 4-gon.

14. ABCD is a 4-gon, and AC, BD intersect at O. Show that a △ having two sides equal to AC, BD respectively, and the contained ∠ equal to ∠ AOB, or ∠ BOC, is equal in area to ABCD.

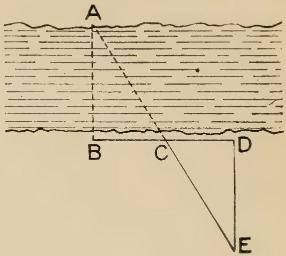
15. Of all the △s that can be formed having two sides respectively equal to two given line-segments, the greatest is that in which the two sides contain a rt. ∠.

16. In a given ||gm inscribe a rhombus having one vertex at a given point in a side of the ||gm.



17. Show from the diagram how the distance between two points A, B at opposite sides of a pond may be found by measurements on land.

18. Show from the diagram how the breadth of a river may be found by measurements made on one side of it.



19. Given a line-segment AB, construct a continuation of it CD, AB and CD being separated by an obstacle.

20. AB, CD are two lines which would meet off the paper. Draw a st. line which would pass through the point of intersection of AB, CD, and bisect the $\angle$ between them.

CHAPTER IV.

Ratio and Proportion.

93. **Definition**:—The **ratio** of one magnitude to another of the same kind is the number of times that the first contains the second; or it is the part, or fraction, that the first magnitude is of the second.

Thus the ratio of one magnitude to another is the same as the measure of the first when the second is taken as the unit.

If a line-segment is 5 cm. in length the ratio of its length to the length of one centimetre is 5, that is, the line-segment is to one centimetre as 5 is to 1.

If two line-segments A, B are respectively 8 inches and 3 inches in length, then the ratio of A to B is 8 to 3.

94. The ratio of one magnitude A to another B is written either $\frac{A}{B}$ or A : B.

The first magnitude is called the **numerator** or **antecedent**; the second the **denominator** or **consequent.**

95. **Definition**:—**Proportion** is the equality of ratios.

The equality of the ratios of A to B and of C to D may be expressed either by $\frac{A}{B} = \frac{C}{D}$, or by A : B :: C : D.

The four magnitudes in a proportion are called **proportionals,** or are said to be **in proportion.**

The first and last are called the **extremes,** and the second and third are called the **means.**

96. Three magnitudes are said to be **in continued proportion,** or in **geometric progression,** when the ratio of the first to the second equals the ratio of the second to the third.

If $\frac{A}{B} = \frac{B}{C}$, A, B and C are in continued proportion.

e.g. :—A = 4 cm., B = 6 cm., C = 9 cm.

The second magnitude of a continued proportion is called the **mean proportional,** or **geometric mean,** of the other two.

97. Two magnitudes of the same kind are **commensurable** when each contains some common measure an integral number of times.

98. Two magnitudes of the same kind are **incommensurable** when there is no common measure, however small, contained in each of them an integral number of times.

The diagonal and side of a square are incommensurable; the ratio of the diagonal to the side being $\sqrt{2} : 1$.

The side of an equilateral triangle and the perpendicular from a vertex to the opposite side are incommensurable; the ratio of a side to the perpendicular being $2 : \sqrt{3}$.

99. The following simple algebraic theorems are used in geometry :—

1. If $\frac{a}{b} = \frac{c}{d}$, $ad = bc$.

If four numbers be in proportion, the product of the extremes is equal to the product of the means.

2. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a}{c} = \frac{b}{d}$.

If four numbers be in proportion, the first is to the third as the second is to the fourth.

When a proportion is changed in this way the second proportion is said to be formed from the first by alternation.

3. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{b} = \frac{c+d}{d}$.

4. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a-b}{b} = \frac{c-d}{d}$.

5. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

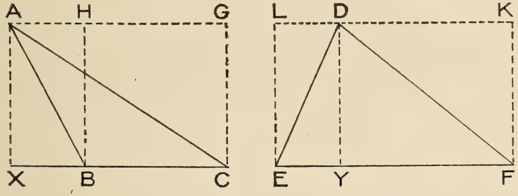
6. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ etc., then each of the equal fractions $= \frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}$

7. If $ad = bc$, $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{c} = \frac{b}{d}$.

Proportion Applied to Triangles and Parallelograms.

100. **Theorem:—Triangles of the same altitude are to each other as their bases.**

Let ABC, DEF be two $\triangle$s of which the $\perp$s AX, DY from A, D to BC, EF respectively are equal to each other.



It is required to prove $\frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF}$.

Let BC, EF contain a, b units of length respectively, and let the common altitude contain c units.

Through A, D draw AHG, LDK $\parallel$ BC, EF respectively. Draw BH, CG $\perp$ BC or AG, and EL, FK $\perp$ EF or LK.

$$\triangle ABC = \tfrac{1}{2} \text{ rect. HC} = \tfrac{1}{2} ac. \qquad (\S\ 73.)$$

$$\triangle DEF = \tfrac{1}{2} \text{ rect. LF} = \tfrac{1}{2} bc.$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} ac}{\frac{1}{2} bc} = \frac{a}{b} = \frac{BC}{EF}.$$

101.—Exercises.

1. **△s on equal bases are to each other as their altitudes.**

2. If two △s are to each other as their bases, their altitudes must be equal.

3. **||gms. of equal altitudes are to each other as their bases.**

4. Construct a △ equal to $\frac{5}{4}$ of a given △.

5. Construct a ||gm. equal to $\frac{5}{2}$ of a given ||gm.

6. ABC, DEF are two △s having AB = DE and ∠ B = ∠ E. Show that △ ABC : △ DEF = BC : EF.

7. If two equal △s be on opposite sides of the same base, the st. line joining their vertices is bisected by the common base, or the base produced.

8. The sum of the ⊥s from any point in the base of an isosceles △ to the two equal sides equals the ⊥ from either end of the base to the opposite side.

9. The difference of the ⊥s from any point in the base produced of an isosceles △ to the equal sides equals the ⊥ from either end of the base to the opposite side.

10. The sum of the ⊥s from any point within an equilateral △ to the three sides equals the ⊥ from any vertex to the opposite side.

11. If st. lines AO, BO, CO be drawn from the vertices of a △ ABC to any point O and AO, produced if necessary, cut BC at D,

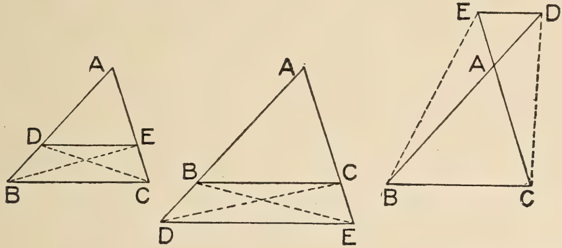
$$\frac{\triangle \text{ AOB}}{\triangle \text{ AOC}} = \frac{\text{BD}}{\text{DC}}.$$

12. In any △ ABC, F is the middle point of AB, E is the middle point of AC, and BE, CF intersect at O. Show that AO produced bisects BC; that is, **the medians of a △ are concurrent.**

13. ABC is a △ and O is any point. AO, BO, CO produced if necessary cut BC, CA, AB at D, E, F respectively. a_1, a_2, b_1, b_2, c_1, c_2 are respectively the numerical measures of BD, DC, CE, EA, AF, FB. Show that $a_1\, b_1\, c_1 = a_2\, b_2\, c_2$. (This is known as Ceva's Theorem.)

102. **Theorem:—A straight line drawn parallel to the sides of a triangle cuts the sides, or the sides produced, proportionally.**

In the $\triangle$ ABC let DE drawn $\parallel$ BC cut AB, AC at D, E respectively.



It is required to show that $\frac{BD}{DA} = \frac{CE}{EA}$.

Join BE, CD.

Then, in any one of the three diagrams, the $\triangle$s BDE, CDE are on the same base DE, and have the same altitude, $\therefore$ $\triangle$ BDE = $\triangle$ CDE; consequently

$$\frac{\triangle\,BDE}{\triangle\,ADE} = \frac{\triangle\,CDE}{\triangle\,ADE}.$$

But $\triangle$s BDE, ADE have the same altitude, viz., the $\perp$ from E to AB,

$$\therefore \frac{\triangle\,BDE}{\triangle\,ADE} = \frac{BD}{DA}. \qquad (\S\ 100.)$$

In the same manner,

$$\frac{\triangle\,CDE}{\triangle\,ADE} = \frac{CE}{EA}.$$

$$\therefore \frac{BD}{DA} = \frac{CE}{EA}.$$

103.—Exercises.

1. E, F are points in AB, AC, the sides of △ ABC, such that EF ∥ BC. Show that AE : AB = AF : AC.

2. The st. line drawn through the middle point of one side of a △ and ∥ to a second side bisects the third side.

3. **If two sides, or two sides produced, of a △ be divided proportionally, the st. line joining the points of division is ∥ to the base.** (Converse of theorem in § 102.)

4. **The st. line joining the middle points of two sides of a △ is ∥ to the base.**

5. The middle points of the sides of a 4-gon are the vertices of a ∥gm.

6. The middle points of the diagonals and the middle points of either pair of opposite sides of a 4-gon are the vertices of a ∥gm.

7. The st. lines joining the middle points of the opposite sides of a 4-gon and the st. line joining the middle points of the diagonals are concurrent.

8. If two sides of a 4-gon be ∥, any st. line drawn ∥ to the ∥ sides and cutting the other sides, will cut these other sides proportionally.

9. ABCD is a 4-gon having AB ∥ DC. P, Q are points in AD, BC respectively such that AP : PD = BQ : QC. Show that PQ ∥ AB or DC.

10. **If two st. lines are cut by a series of ∥ st. lines, the intercepts on one are proportional to the corresponding intercepts on the other.**

11. D, E are points in AB, AC, the sides of △ ABC, such that DE ∥ BC; BE, CD meet at F. Show that △ ADF = △ AEF.

Show also that AF bisects DE and BC.

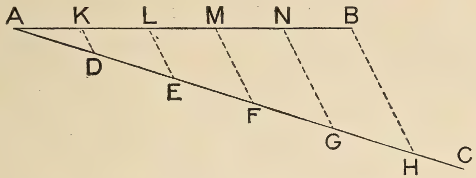
12. Through D, any point in the side BC of △ ABC, DE, DF are drawn ∥ AB, AC respectively and meeting AC, AB at E, F. Show that △ AEF is a mean proportional between △s FBD, EDC.

13. ACB, ADB are two △s on the same base AB. E is any point in AB. EF is ∥ AC and meets BC at F. EG is ∥ AD and meets BD at G. Prove FG ∥ CD.

14. D is a point in the side AB of △ ABC; DE is drawn ∥ BC and meets AC at E; EF is drawn ∥ AB and meets BC at F. Show that AD : DB = BF : FC.

104. **Problem :—To divide a given line-segment into any number of equal parts.**

Let AB be the given line-segment.

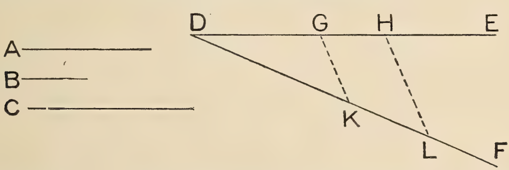


It is required to divide AB into any number of equal parts (five for example).

Through A draw any st. line AC, and with the dividers set at any convenient distance mark off in succession from AC the required number of equal parts AD, DE, EF, FG, GH. Join HB. Through D, E, F, G draw lines || BH cutting AB at K, L, M, N. Then AB is equally divided at K, L, M and N. [*The proof is left for the pupil.*]

105. **Problem :—To find a fourth proportional to three given line-segments taken in a given order.**

Let A, B, C, be the three line-segments.



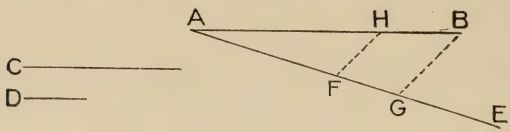
It is required to find a fourth proportional to A, B and C.

From a point D draw two st. lines DE, DF. From DE cut off in succession DG = A and GH = B. From DF cut off DK = C. Join GK. Through H draw HL || GK and meeting DF in L.

Then A : B = C : KL, or KL is the fourth proportional to A, B and C. [*The proof is left for the pupil.*]

106. **To divide a given line-segment in the ratio of two other given line-segments.**

Let AB, C and D be three given line-segments.



It is required to divide AB into two parts in the ratio of C to D.

Through A draw any st. line AE. From AE cut off in succession AF = C and FG = D. Join BG. Through F draw FH ∥ GB.

Then AB is divided at H, so that AH : HB = C : D.

[*The proof is left for the pupil.*]

107.—Exercises.

1. Divide the area of a given △ into parts that are in the ratio of two given line-segments.

2. Divide the area of a ∥gm into parts that are in the ratio of two given line-segments.

3. Find a third proportional to two given line-segments. Show how two third proportionals, one greater than either of the given line-segments and the other less than either, may be found.

4. Find a point H in a given line-segment AB produced such that AH : HB equals the ratio of two given line-segments.

5. BAC is a given ∠ and P is a given point. Through P draw a st. line DPE cutting AB at D and AC at E such that DP : PE equals the ratio of two given line-segments.

6. Draw a line-segment AB, and on it mark two points P, Q. Divide another given line-segment CD similarly to AB.

7. If AB, C and D be three given line-segments, show that there is only one point H in AB such that $\frac{AH}{HB}=\frac{C}{D}$.

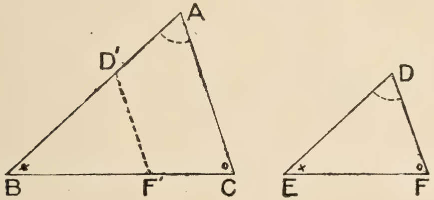
Show also that there is only one point K in AB produced such that $\frac{AK}{KB}=\frac{C}{D}$.

Similar Triangles.

108. **Definition**:—If the three angles of one triangle be respectively equal to the three angles of another triangle, the triangles are said to be **similar.**

109. **Theorem:—If two triangles are similar, their corresponding sides are proportional.**

Let ABC, DEF be two $\triangle$s having $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E and consequently $\angle$ C = $\angle$ F.



It is required to prove $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$.

Apply $\triangle$ DEF to $\triangle$ ABC so that E falls on B and EF falls along BC. Then because $\angle$ E = $\angle$ B, ED falls along BA and $\triangle$ DEF takes the position D′BF′.

$$\angle D'F'B = \angle C, \therefore D'F' \parallel AC$$

and consequently $\dfrac{AD'}{D'B} = \dfrac{CF'}{F'B}$. (§ 102.)

Hence $\dfrac{AD' + D'B}{D'B} = \dfrac{CF' + F'B}{F'B}$; (§ 99, 3.)

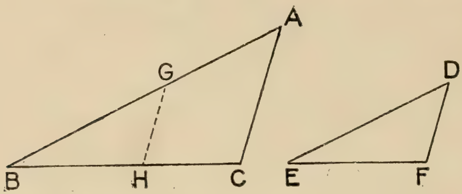
that is, $\dfrac{AB}{DE} = \dfrac{BC}{EF}$.

In the same manner it may be shown that $\dfrac{BC}{EF} = \dfrac{CA}{FD}$.

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

110. **Theorem :—If the sides of one triangle be proportional to the sides of another, the triangles are similar, the equal angles being opposite to pairs of corresponding sides.** (Converse of theorem in § 109.)

Let ABC, DEF be two $\triangle$s such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$; AB, BC, CA corresponding to DE, EF, FD respectively.



It is required to show that $\angle$ C = $\angle$ F, $\angle$ A = $\angle$ D and $\angle$ B = $\angle$ E.

From BC cut off BH = EF. Through H draw HG CA cutting BA at G.

HG is $\parallel$ CA, $\therefore$ $\angle$ BHG = $\angle$ C and $\angle$ BGH = $\angle$ A, $\therefore$ $\triangle$ BGH is similar to $\triangle$ BAC and consequently

$$\frac{AB}{BG} = \frac{BC}{BH} = \frac{CA}{HG}; \qquad (\S\ 109.)$$

but, by hypothesis $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$;

$\therefore$, by division, $\frac{BG}{DE} = \frac{BH}{EF} = \frac{HG}{FD}$.

But BH was made = EF, and $\therefore$ BG = DE and HG = FD.

Then in $\triangle$s GBH, DEF,

BG = DE, BH = EF, and HG = FD,

$\therefore$ $\angle$ B = $\angle$ E, $\angle$ BHG = F, and $\angle$ BGH $\angle$ D.

But $\angle$ C = $\angle$ BHG, and $\angle$ A = $\angle$ BGH,

$\therefore$ $\angle$ C = $\angle$ F, and $\angle$ A = $\angle$ D.

111.—Exercises.

1. The st. line joining the middle points of the sides of a $\triangle$ is $\parallel$ to the base, and equal to half of it.

2. If two sides of a 4-gon be $\parallel$, the diagonals cut each other proportionally.

3. In the $\triangle$ ABC the medians BE, CF cut at G. Show that BG = twice GE and CG = twice GF.

4. Using the theorem in Ex. 3, devise a method of trisecting a st. line-segment.

5. If the diagonals of a 4-gon divide each other proportionally, one pair of sides are $\parallel$.

6. In similar $\triangle$s $\perp$s from corresponding vertices to the opposite sides are in the same ratio as the corresponding sides.

7. In similar $\triangle$s the bisectors of two corresponding $\angle$s, terminated by the opposite sides, are in the same ratio as the corresponding sides.

8. ABCD is a $\parallel$gm, and a line through A cuts BD at E, BC at F and meets DC produced at G. Show that AE : EF = AG : AF.

9. If two $\parallel$ line-segments, AB, CD be divided at E, F respectively so that AE : EB = CF : FD, then AC, BD and EF are concurrent.

10. The median drawn to a side of a $\triangle$ bisects all st. line-segments $\parallel$ to that side and terminated by the other two sides, or those sides produced.

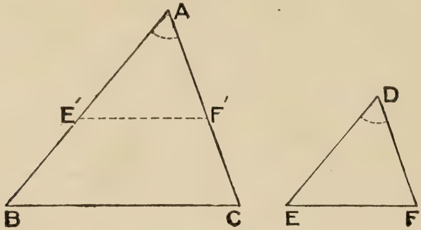
11. ABCD is a $\parallel$gm. AD is bisected at E and BC at F. Show that AF and CE trisect the diagonal BD.

12. If the st. lines OAB, OCD, OEF be similarly divided, the $\triangle$s ACE, BDF are similar.

13. If the corresponding sides of two similar $\triangle$s be $\parallel$, the st. lines joining the corresponding vertices are concurrent.

112. **Theorem :—If two triangles have two sides of one proportional to two sides of the other and the angles contained by these sides equal to each other, the triangles are similar, and the equal angles are opposite to the corresponding sides.**

Let ABC, DEF be two $\triangle$s having AB : DE = AC : DF and $\angle$ A = $\angle$ D.



It is required to show that $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F.

Apply the $\triangle$ DEF to $\triangle$ ABC so that D falls on A and DE falls along AB, then because $\angle$ D = $\angle$ A, DF will fall along AC and the $\triangle$ DEF takes the position AE′F′.

$$\because \frac{AB}{AE'} = \frac{AC}{AF'}$$

$$\therefore \frac{AB - AE'}{AE'} = \frac{AC - AF'}{AF'}. \qquad (\S\ 99,\ 4.)$$

that is $\dfrac{BE}{AE'} = \dfrac{CF'}{AF'}$

and $\therefore$ E′F′ is $\parallel$ BC. (Ex. 3, § 103.)

$\therefore$ $\angle$ AE′F′ = $\angle$ B and $\angle$ AF′E′ = $\angle$ C, and consequently $\triangle$ DEF is similar to $\triangle$ ABC.

$$\text{Also } \frac{BC}{EF} = \frac{AB}{DE} \text{ or } \frac{AC}{DF}.$$

113. **Theorem :—If two triangles have two sides of one proportional to two sides of the other, and the angles opposite one pair of corresponding sides in the proportion equal to each other, the**

angles opposite the other pair of corresponding sides in the proportion are either equal or supplementary.

Let ABC, DEF be two $\triangle$s having AB : DE = AC : DF and $\angle$ B = $\angle$ E.

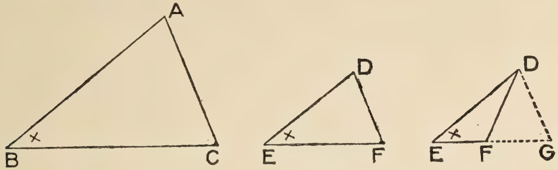


FIG. 1 FIG. 2

It is required to show that either $\angle$ C = $\angle$ F, or that $\angle$ C + $\angle$ DFE = two rt. $\angle$s.

Case I :—If $\angle$ A = $\angle$ D (Fig. 1).

Then in $\triangle$s ABC, DEF, $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E, and $\therefore$ $\angle$ C = $\angle$ F.

In this case the $\triangle$s are similar.

Case II :—If $\angle$ A be not equal to $\angle$ D (Fig. 2).

At D make $\angle$ EDG = $\angle$ A and produce the arm of this $\angle$ to meet EF, or EF produced, at G.

Then in $\triangle$s ABC, DEG,

$\angle$ A = $\angle$ EDG, $\angle$ B = $\angle$ E and $\therefore$ $\angle$ C = $\angle$ G;

consequently $\dfrac{AC}{DG} = \dfrac{AB}{DE}$; (§ 109.)

but, by hypothesis, $\dfrac{AC}{DF} = \dfrac{AB}{DE}$,

$\therefore \dfrac{AC}{DG} = \dfrac{AC}{DF}$ and hence DF = DG.

In $\triangle$ DFG, DF = DG, $\therefore$ $\angle$ DFG = $\angle$ G.

But it has been shown that $\angle$ C = $\angle$ G,

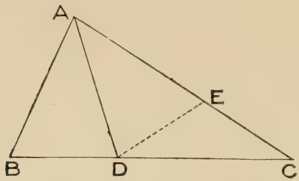
$\therefore$ $\angle$ DFG = $\angle$ C.

$\angle$ DFE + $\angle$ DFG = two rt. $\angle$s,

$\therefore$ $\angle$ DFE + $\angle$ C = two rt. $\angle$s.

114. **Theorem:—The bisector of the vertical angle of a triangle divides the base into segments proportional to the sides of the triangle.**

Let the bisector of the $\angle$ A of the $\triangle$ ABC cut BC at D.



It is required to show that $\dfrac{BD}{DC} = \dfrac{AB}{AC}$.

From AC cut off AE = AB, and join DE.

In $\triangle$s ABD, EAD,

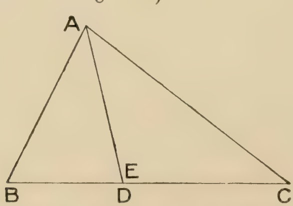
AB = AE, AD is common, and $\angle$ BAD = $\angle$ EAD,

$$\therefore \triangle ABD = \triangle ADE.$$

Then $\dfrac{BD}{DC} = \dfrac{\triangle ABD}{\triangle ADC} = \dfrac{\triangle ADE}{\triangle ADC} = \dfrac{AE}{AC} = \dfrac{AB}{AC}$;

that is, $\dfrac{BD}{DC} = \dfrac{AB}{AC}$.

115. **Theorem:—If the base of a triangle be divided into segments proportional to the sides of the triangle, the straight line joining the point of section to the vertex bisects the vertical angle.** (Converse of Theorem in § 114.)



Let D be a point in the base BC of $\triangle$ ABC, such that BD : DC = AB : AC.

It is required to show that AD bisects $\angle$ BAC.

Bisect $\angle$ BAC and let the bisector cut BC at E.

$$\text{Then } \frac{BE}{EC} = \frac{BA}{AC}, \qquad (\S\ 114.)$$

$$\text{But, by hypothesis } \frac{BD}{DC} = \frac{BA}{AC};$$

$$\therefore \frac{BE}{EC} = \frac{BD}{DC},$$

$$\text{and } \frac{BE+EC}{EC} = \frac{BD+DC}{DC}, \qquad (\S\ 99,\ 3.)$$

$$\text{that is, } \frac{BC}{EC} = \frac{BC}{DC},$$

$\therefore$ EC = DC, and the point E must coincide with D.

Consequently AD is the bisector of $\angle$ A.

116.—Exercises.

1. The sides of a $\triangle$ are 4 cm., 5 cm., 6 cm. Calculate the lengths of the segments of each side made by the bisector of the opposite $\angle$.

2. AD bisects $\angle$ A of $\triangle$ ABC and meets BC at D. Find BD and CD in terms of a, b, and c.

3. **If the bisector of the exterior $\angle$ at the vertex of a $\triangle$ cuts the base produced, the distances of the point of section from the ends of the base are proportional to the sides of the $\triangle$.**

4. **If a point be taken in the base produced of a $\triangle$, such that its distances from the ends of the base are proportional to the sides, the line joining this point to the vertex bisects the exterior vertical $\angle$ of the $\triangle$.**

5. In $\triangle$ ABC, $a = 7$, $b = 5$, $c = 3$. The bisectors of the exterior $\angle$s at A, B, C meet BC, CA, AB respectively at D, E, F. Calculate BD, AE and AF.

6. In $\triangle$ ABC, the bisector of the exterior vertical $\angle$ at A meets BC produced at D. Find BD and CD in terms of a, b and c.

7. AD is a median of △ ABC; ∠s ADB, ADC are bisected by DE, DF meeting AB, AC at E, F respectively. Prove EF ∥ BC.

8. The bisectors of ∠s A, B, C in △ ABC meet BC, CA, AB at D, E, F respectively. Show that AF.BD.CE = FB.DC.EA.

9. If the bisectors of ∠s A, C in the 4-gon ABCD meet in the diagonal BD, the bisectors of ∠s B, D meet in the diagonal AC.

10. If the bisectors of ∠s ABC, ADC meet at a point in AC, the bisectors of the exterior ∠s at B and D meet in AC produced.

11. In similar △s medians drawn from corresponding vertices and terminated by the opposite sides are proportioned to the corresponding sides.

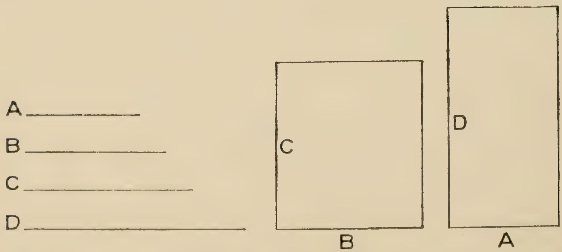
12. In a △ ABC, AD is drawn ⊥ BC. If BD : DA = DA : DC, prove that BAC is a rt. ∠.

13. In the corresponding sides BC, EF of the similar △s ABC, DEF the points G, H are taken such that BG : GC = EH : HF. Prove AG : DH = BC : EF.

Proportion Applied to Areas.

117. **Theorem:—If four straight line-segments are proportionals, the rectangle contained by the means equals the rectangle contained by the extremes.**

Let A, B, C, D be four line-segments such that A : B = C : D.



It is required to show that the rect. contained by B and C equals the rect. contained by A and D.

Let a, b, c, d be the numerical measures of A, B, C, D respectively.

$$\text{Then } \frac{a}{b} = \frac{c}{d}$$

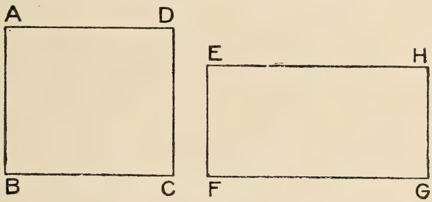
$$\text{and } \therefore\ bc = ad;$$

but bc is the numerical measure of rect. B.C, and ad is the numerical measure of A.D.

$$\therefore \text{ rect. B.C} = \text{rect. A.D.}$$

118. **Theorem:—If two rectangles be equal to each other, the length of one is to the length of the other as the breadth of the second is to the breadth of the first.**

Let ABCD, EFGH be two equal rectangles.



It is required to show that $\dfrac{BC}{FG} = \dfrac{EF}{AB}$.

Let a, b, c, d be the numerical measures of BC, BA, FG, EF respectively.

Then since the rectangles are equal,

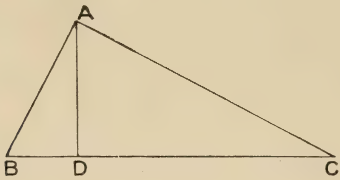
$$ab = cd,$$

$$\text{and } \therefore\ \frac{a}{c} = \frac{d}{b};$$

$$\text{that is, } \frac{BC}{FG} = \frac{EF}{AB}.$$

119. **Theorem :—In any right-angled triangle, if a perpendicular be drawn from the vertex of the right angle to the hypotenuse, (*a*) the square on the perpendicular equals the rectangle contained by the segments of the hypotenuse, (*b*) the square on one side of the right-angled triangle equals the rectangle contained by the hypotenuse and the segment adjacent to that side.**

Let ABC be a rt.-$\angle$d $\triangle$, and AD the $\perp$ to the hypotenuse BC from A.



It is required to show that the square on AD = rect. BD.DC.

In $\triangle$s ABD, ABC, $\angle$ ADB = $\angle$ BAC and $\angle$ B is common,

$$\therefore \angle \text{ BAD} = \angle \text{ BCA}.$$

Similarly, $\angle$ DAC = $\angle$ B.

$\therefore$ the three $\triangle$s ABD, ADC, BAC are similar.

From the similar $\triangle$s BAD, ADC;

$$\frac{BD}{DA} = \frac{DA}{DC}$$

and $\therefore$ DA^2 = rect. BD.DC. (§ 117.)

It is required to show that the square on AB = rect. BC.BD.

From the similar $\triangle$s ABD, ABC

$$\frac{BC}{BA} = \frac{BA}{BD},$$

$$\text{and} \therefore BA^2 = \text{rect. } BC.BD.$$

120.—Exercises.

1. In any two equal $\triangle$s ABC, DEF, if AG, DH be $\perp$s to BC, EF respectively, AG : DH = EF : BC.

2. In any $\triangle$ the $\perp$s from the vertices to the opposite sides are inversely as the sides.

3. In the diagram of § 119, show that rect. AD.BC = rect. BA.AC. Give a general statement of this theorem.

4. ABC, DEF are two equal $\triangle$s having also $\angle B = \angle E$. Show that $\frac{BC}{EF} = \frac{DE}{AB}$.

5. ABCD, EFGH are two equal ||gms having also $\angle B = \angle F$. Show that $\frac{BC}{FG} = \frac{FE}{BA}$.

6. ABCD is a given rect. and EF a given line-segment. It is required to make a rect. equal in area to ABCD and having one of its sides equal to EF.

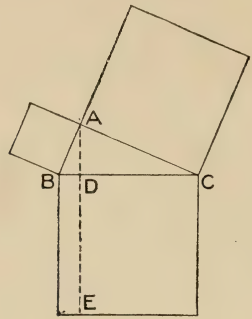
7. Make a rect. equal in area to a given $\triangle$ and having one of its sides equal to a given line-segment.

8. Show how to construct a rect. equal in area to a given polygon and having one of its sides equal to a given line-segment.

---

121. **Theorem:—The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides.**

Let ABC be a rt.-∠d △ having A the rt. ∠ and having squares described on the three sides.



It is required to show that sq. on BC = sq. on BA + sq. on AC.

Draw AD ⊥ BC and produce AD to cut the opposite side of sq. on BC at E.

ABD, ABC are similar △s,

$\therefore \dfrac{BC}{BA} = \dfrac{BA}{BD}$, and consequently sq. on BA = rect. BC.BD = rect. BE.

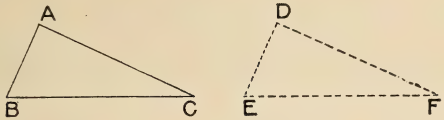
Similarly, sq. on AC = rect. CE.

Then sq. on BC = rect. BE + rect. CE.

= sq. on BA + sq. on AC.

122. **Theorem:—If the square on one side of a triangle be equal to the sum of the squares on the other two sides, the angle contained by these sides is a right angle.**

Let ABC be a △ having the sq. on BC = sq. on AB + sq. on AC.



It is required to show that ∠ A is a rt. ∠.

Make a rt.-∠ D and cut off DE = AB and DF = AC. Join EF.

The sq. on BC = sq. on BA + sq. on AC.

= sq. on ED + sq. on DF.

= sq. on EF. (§ 121.)

∴ EF = BC.

Then in △s ABC, DEF, AB = DE, AC = DF and BC = EF,

∴ ∠ A = ∠ D = a rt. ∠.

123.—Exercises.

1. Construct a square equal to the sum of two given squares.

2. Construct a square equal to twice a given square.

3. The sq. on the diagonal of a given square is twice the given square.

4. Construct a square equal to the sum of three given squares.

5. Construct a square equal to five times a given square.

6. Construct a square equal to the difference of two given squares.

7. Construct a square equal to half a given square.

8. Draw line-segments $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ times a given line-segment.

9. ABC is a △, and AD ⊥ BC. Show that the differences of squares on BD, DC equals the difference of squares on AB, AC.

10. A is a given line-segment. Find another line-segment B, such that the difference of the squares on A and B may be equal to the difference of two given squares.

11. If the diagonals of a quadrilateral cut at rt.-$\angle$s, the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair.

12. The sum of the squares on the diagonals of a rhombus equals the sum of the squares on the four sides.

13. Five times the square on the hypotenuse of a rt.-$\angle$d $\triangle$ equals four times the sum of the squares on the medians drawn to the other two sides.

14. In an isosceles rt.-$\angle$d $\triangle$ the sides have the ratios $1 : 1 : \sqrt{2}$.

15. If the angles of a $\triangle$ be 90°, 30°, 60°, the sides have the ratios $2 : 1 : \sqrt{3}$.

16. Divide a st. line-segment into two parts such that the sum of the squares on the parts equals the square on another given line-segment. When is this impossible?

17. In the line-segment AB produced find a point C such that the sum of the squares on AC, BC equals the square on a given line-segment.

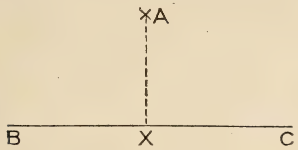
18. Divide a given line-segment into two parts such that the square on one part is double the square on the other part.

19. ABCD is a rect., and P is any point. Show that $PA^2 + PC^2 = PB^2 + PD^2$.

20. ABC is a $\triangle$ rt.-$\angle$d at A. E is a point on AC and F is a point on AB. Show that $BE^2 + CF^2 = EF^2 + BC^2$.

124. **Definition** :—If a perpendicular be drawn from a given point to a given straight line, the foot of the perpendicular is said to be the **projection of the point on the line.**

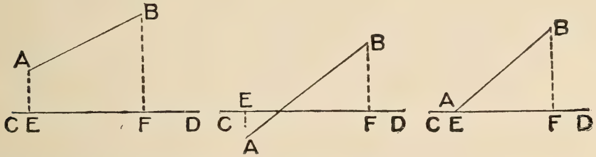
From the point A the ⊥ AX is drawn to the line BC.



The point X is the projection of the point A on the st. line BC.

125. **Definition** :—If from the ends of a given line-segment perpendiculars be drawn to a given straight line, the segment intercepted on the straight line is called the **projection of the line-segment on the straight line.**

AB is a line-segment and CD a st. line. AE, BF are drawn ⊥ CD.



EF is the projection of AB on CD.

126.—Exercises.

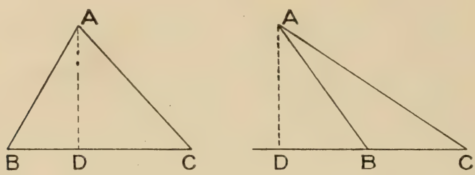
1. Show that a line-segment is never less than its projection on a st. line. In what case are they equal? In what case is the projection of a line-segment on a st. line just a point?

2. ABC is a △ having $a = 36$ mm., $b = 40$ mm. and $c = 45$ mm. Draw the △ and measure the projection of AB on BC. (*Ans.* 23.9 mm. nearly.)

3. ABC is a △ having $a = 5$ cm., $b = 7$ cm., $c = 10$ cm. Draw the △ and measure the projection of AB on BC. (*Ans.* 76 mm.)

127. **Theorem :—In any triangle the square on the side opposite an acute angle is equal to the sum of the squares on the sides which contain the acute angle diminished by twice the rectangle contained by either of these sides and the projection on that side of the other.**

Let ABC be a △ having C an acute ∠ and CD the projection of CA on CB.



It is required to show that the sq. on AB = sq. on AC + sq. on BC − twice rect. BC.CD.

ABD is a rt.-∠d △,

∴ sq. on AB = sq. on BD + sq. on AD. (§ 121.)

BD is the difference between BC and CD, ∴ sq. on BD = sq. on BC + sq. on CD − twice rect. BC.CD. (§ 88.)

∴ sq. on AB = sq. on AD + sq. on DC + sq. on BC − twice rect. BC.CD.

But ADC is a rt.-∠d △, and ∴ sq. on AD + sq. on DC = sq. on AC.

And consequently sq. on AB = sq. on AC + sq. on BC − twice rect. BC.CD.

128.—Exercises.

1. **In an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides that contain the obtuse angle increased by twice the rectangle contained by either of these sides and the projection on that side of the other.**

2. ABC is a △ having C an ∠ of 60°. Show that sq. on AB = sq. on BC + sq. on AC − rect. BC.AC.

3. ABC is a △ having C an ∠ of 120°. Show that sq. on AB = sq. on BC + sq. on AC + rect. BC.AC.

4. ABC is a △, CD the projection of CA on CB, and CE the projection of CB on CA. Show that rect. BC.CD = rect. AC.CE.

5. **In any △ the sum of the squares on the sides equals twice the square on half the base together with twice the square on the median drawn to the base.**

6. In any quadrilateral the sum of the squares on the four sides exceeds the sum of the squares on the diagonals by four times the square on the st. line joining the middle points of the diagonals.

What does this proposition become when the quadrilateral is a ||gm.

7. ABC is a △ having a = 47 mm., b = 62 mm., and c = 84 mm. D, E, F are the middle points of BC, CA, AB respectively. Calculate the lengths of AD, BE and CF. Test your results by drawing and measurement.

8. The squares on the diagonals of a 4-gon are together double the sum of the squares on the line-segments joining the middle points of opposite sides.

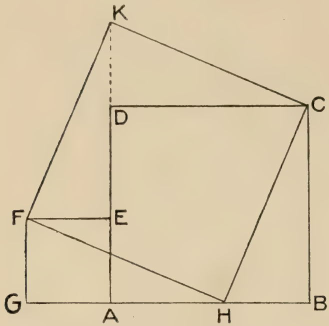
9. If the medians of a △ intersect at G,

$$AB^2 + BC^2 + CA^2 = 3\,(GA^2 + GB^2 + GC^2).$$

10. C is the middle point of a line-segment AB. P is any point on the circumference of a circle of which C is the centre. Show that $PA^2 + PB^2$ is constant.

11. Two circles have the same centre. Prove that the sum of the squares of the distances from any point on the circumference of either circle to the ends of the diameter of the other is constant.

12. The square on the base of an isosceles △ is equal to twice the rect. contained by either of the equal sides and the projection on it of the base.



13. Prove the proposition in § 121 from the following construction:—

Draw two squares, ABCD, AEFG, having AD, AE in the same st. line.

Cut off GH and EK each = AB.

Join FH, HC, CK, KF.

14. If two sides of a △ be unequal, the median drawn to the shorter side is greater than the median drawn to the longer side.

15. If, from any point P within △ ABC, ⊥s PX, PY, PZ be drawn to BC, CA, AB respectively,

$$BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2.$$

16. D, E, F are the middle points of BC, CA, AB respectively in △ ABC. Prove that

$$3(AB^2 + BC^2 + CA^2) = 4(AD^2 + BE^2 + CF^2).$$

17. G is the centroid of △ ABC, and P is any point. Show that

$$PA^2 + PB^2 + PC^2 = AG^2 + BG^2 + CG^2 + 3\,PG^2.$$

18. Find the point P in the plane of the △ ABC such that the sum of the squares on PA, PB, PC may be the least possible.

19. Check the results in Exs. 2 and 3, § 126, by calculation.

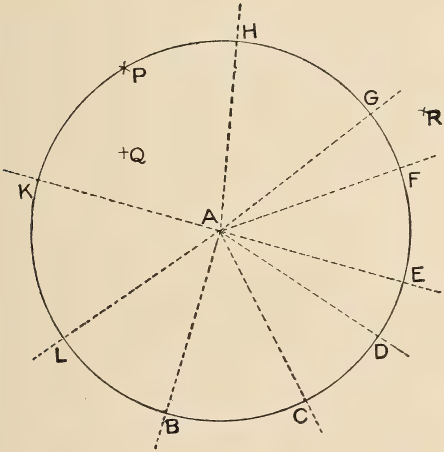
20. If, in the diagram of § 127, the ∠ C becomes more and more acute and finally the point A comes down to the line BC, what does the theorem become?

21. If, in Ex. 1, § 128, the obtuse ∠ becomes greater and greater and finally becomes a st. ∠, what does the theorem become?

CHAPTER V.

LOCI.

129. **Example 1.**—A is a point and from A straight lines are drawn in different directions in the same plane.



On each line a distance of one inch is measured from A and the resulting points are B, C, D, etc.

Is there any one line that contains all of the points in the plane that are at a distance of one inch from A?

To answer this question describe a circle with centre A and radius one inch. The circumference of this circle is a line that passes through all the points.

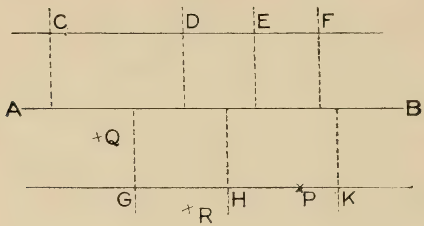
Mark any other point P on the circumference. What is the distance of P from A? From the nature of a circle the answer to this question is one inch.

If any point Q be taken within the circle, its distance from A is less than one inch, and if any point R be taken without the circle, its distance from A is greater than one inch.

Thus every point in the circumference satisfies the condition of being just one inch from A, and no point, in the plane, that is not on the circumference does satisfy this condition.

This circumference is called the **locus** of all points in the plane that are at a distance of one inch from A.

Example 2:—AB is a straight line of indefinite length, to which any number of perpendiculars are drawn.



On each of these perpendiculars a distance of one centimetre is measured from AB, and the resulting points are C, D, E, etc.

Are there any lines that contain all of the points, such as C, D, etc., that are at a distance of one centimetre from AB?

Draw two straight lines parallel to AB, each at a distance of one centimetre from AB, and one or other of these lines will pass through each of the points.

Any point P in CF, or in GK, is at a distance of one centimetre from AB; any point Q in the space between

CF and GK is less than one centimetre from AB, and any point R in the plane and neither between CF and GK nor in one of these lines is more than one centimetre from AB.

Thus every point in CF and GK satisfies the condition of being just one centimetre from AB, and no point outside of these lines and in the plane does satisfy this condition.

The two lines CF, GK make up the **locus** of all points in the plane that are at a distance of one centimetre from AB.

130. **Definition** :—When a figure consisting of a line or lines contains all the points that satisfy a given condition, and no others, this figure is called the **locus** of these points.

131. In place of speaking of the "locus of the points which satisfy a given condition" the alternative expression "locus of the point which satisfies a given condition" may be used.

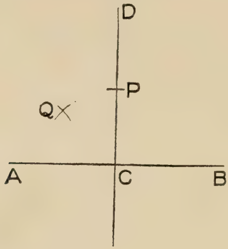
Suppose a point to move in a plane so that it traces out a continuous line, but its distance from a fixed point A in the plane is always one inch; then it must move on the circumference of the circle of centre A and radius one inch, and the locus of the point in its different positions is that circumference.

The following definition of a locus may thus be given as an alternative to that in § 130.

Definition :—If a point moves on a line, or on lines, so that it constantly satisfies a given condition, the figure consisting of the line, or lines, is the **locus** of the point.

132. **Problem :—To find the locus of the point that moves so as to be always equally distant from two given points.**

Let A, B be the two given points.



Join AB and bisect AB at C.

C is a point on the locus.

Through C draw a st. line CD $\perp$ AB.

The pupil may easily show that (*a*) any point P in CD is equally distant from A and B, and (*b*) that any point Q, not in CD is nearer to one of the points C, D than it is to the other.

Thus if the point moves on CD it will be constantly at equal distances from A and B, and if it leaves CD it will no longer be equally distant from the points.

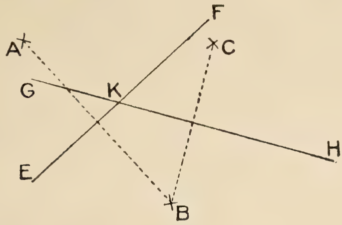
CD then is the required locus.

133. **Problem :—To find the point that is equally distant from three given points, that are not in the same straight line.**

Let A, B, C be the three given points.

It is required to find a point equally distant from A, B and C.

Draw EF the locus of all points that are equally distant from A and B (§ 132).



Draw GH the locus of all points that are equally distant from B and C.

Let EF and GH meet at K.

Then K is the required point.

K is on EF, $\therefore$ KA = KB.

K is on GH, $\therefore$ KB = KC.

Consequently K is equally distant from A, B and C.

134.—Exercises.

1. Find the locus of the centres of all circles that pass through two given points.

2. Describe a circle to pass through two given points and have its centre in a given st. line.

3. Describe a circle to pass through two given points and have its radius equal to a given line-segment. Show that generally two such circles may be described. When will there be only one?

4. Show that the locus of a point which moves so that it is always equidistant from two given intersecting st. lines consists of two st. lines at rt. $\angle$s to each other.

5. In a given st. line find two points each of which is equally distant from two given intersecting st. lines.

When will there be only one solution?

6. Find the locus of the vertices of all $\triangle$s on a given base which have the medians drawn to the base equal to a given line-segment.

7. Find the locus of the vertices of all $\triangle$s on a given base which have one side equal to a given line-segment.

8. Construct a $\triangle$ having given the base, the median drawn to the base, and the length of one side.

9. Find the locus of the vertices of all $\triangle$s on a given base which have a given altitude.

10. Construct a $\triangle$ having given the base, the median drawn to the base, and the altitude.

11. Construct a $\triangle$ having given the base, the altitude and one side.

12. Find the locus of a point such that the sum of the squares on its distances from two given points is constant.

13. Find the locus of a point such that the difference of the squares on its distances from two given points is constant.

14. Find the locus of a point such that the sum of its distances from two given intersecting st. lines is equal to a given line-segment.

15. Find the locus of a point such that the difference of its distances from two given intersecting st. lines is equal to a given line-segment.

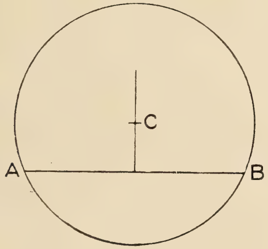
16. Find the locus of the vertices of all $\triangle$s on a given base which have the median drawn from one end of the base equal to a given line-segment.

17. Show that, if the ends of a st. line-segment of constant length slide along two st. lines at rt. $\angle$s to each other, the locus of its middle point is a circle.

CHORDS OF A CIRCLE.

135. **Definition** :—A **circle** is a figure consisting of one closed curved line, called the circumference, and is such that all straight lines drawn from a certain point within the figure, called the centre, to the circumference are equal to each other.

136. The locus of all points equally distant from the ends of a st. line-segment is the st. line which bisects the line-segment at rt. ∠s (§ 132), and the centre of a circle is a point equally distant from the ends of any chord of the circle; hence the three following theorems are true :—



(*a*) **The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.**

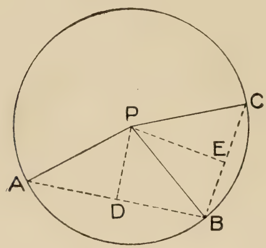
(*b*) **The straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to the chord.**

(*c*) **The straight line bisecting a chord of a circle at right angles passes through the centre of the circle.**

As an exercise the pupil should give independent proofs of theorems (*a*), (*b*) and (*c*).

137. **Theorem:—If from a point within a circle more than two equal straight line-segments be drawn to the circumference, that point is the centre.**

Let P be a point within the circle ABC such that PA, PB, PC are equal to each other.



It is required to show that P is the centre of the circle.

Join AB, BC and from P draw PD, PE $\perp$ AB, BC respectively.

Since P is equally distant from A and B, it must be in the st. line bisecting AB at rt. $\angle$s (§ 132); $\therefore$ PD bisects AB and consequently (§ 136, *c*), the centre of the circle is somewhere in the line PD.

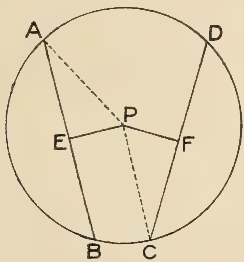
In the same manner it may be shown that the centre is somewhere in PE.

But P is the only point common to PD and PE.

$\therefore$ P is the centre.

138. **Theorem :—Chords that are equally distant from the centre of a circle are equal to each other.**

Let ABC be a circle of which P is the centre and let AB, CD be two chords such that the ⊥s PE, PF from P to AB, CD respectively are equal.



It is required to show that AB = CD.

Join PA, PC.

In the △s PAE, PCF,

AP = CP, PE = PF and ∠s PEA, PFC are rt. ∠s.

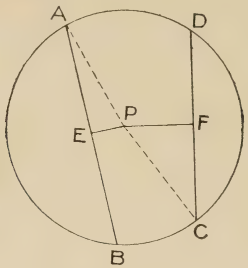
∴ AE = CF. (§ 66, Ex. 1.)

But AB = twice AE and CD = twice CF. (§ 136, *a.*)

∴ AB = CD.

139. **Theorem :—Of two chords in a circle the one which is nearer to the centre is greater than the one which is more remote from the centre.**

Let P be the centre of a circle ABC, and AB, CD two chords such that PE, the distance of AB from the centre, is less than PF, the distance of CD from the centre.



It is required to show that AB > CD.

Join PA, PC.

In the $\triangle$ PAE, $\angle$ PEA is a rt. $\angle$,

$\therefore$ sq. on PA = sq. on PE + sq. on AE. (§ 121.)

In the same manner,

sq. on PC = sq. on PF + sq. on FC.

But sq. on PA = sq. on PC;

$\therefore$ sq. on PE + sq. on AE = sq. on PF + sq. on FC.

And since PE < PF; sq. on PE < sq. on PF;

$\therefore$ sq. on AE > sq. on CF,

and AE > CF.

But AB = twice AE, and CD = twice CF;

$\therefore$ AB > CD.

140.—Exercises.

1. Circumscribe a circle about a given $\triangle$.

2. Find the centre of a given circle.

3. **If two chords of a circle be equal to each other, they are equally distant from the centre.**

4. **If two chords of a circle be unequal, the greater is nearer to the centre than the less.**

5. Through a given point within a circle draw a chord that is bisected at that point.

6. Through a given point within a circle draw the shortest chord.

7. **In a circle any chord that does not pass through the centre is less than a diameter.**

8. A chord a inches long is placed in a circle of radius b inches. Find an algebraic expression for the distance of the chord from the centre.

9. ACB is a diameter, and C the centre of a circle. D is any point on AB, or on AB produced, and P is any point except A or B on the circumference. Show that DP is intermediate in magnitude between DA and DB.

10. O is the centre of a circle, and P is any point. If two st. lines be drawn through P, cutting the circle, and making equal $\angle$s with PO, the chords intercepted on these lines by the circumference are equal to each other.

11. O is the centre of a circle, and P is any point. On two lines drawn through P chords AB, CD are intercepted by the circumference. If $\angle$ BPO $>$ $\angle$ DPO, the chord AB $<$ chord CD.

12. Describe a circle with a given centre to cut a given circle at the extremities of a diameter.

13. If each of two equal st. line-segments have one extremity on one of two concentric circles, and the other extremity on the other, the line-segments subtend equal $\angle$s at the common centre.

14. **If two circles cut each other, the st. line joining their centres bisects their common chord at rt.-$\angle$s.**

15. The locus of the middle points of a system of $\parallel$ chords in a circle is a diameter of the circle.

16. From any point in a circle which is not the centre equal st. lines can be drawn to the circumference only in pairs.

Angles in a Circle.

141. **Theorem :—The angle which an arc of a circle subtends at the centre is double the angle which it subtends at any point on the remaining part of the circumference.**

Let ABC be an arc of a circle, D the centre, and E any point on the remaining part of the circumference,

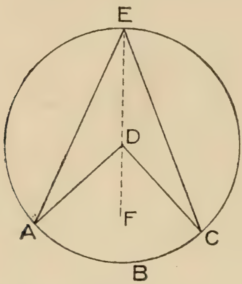


Fig. 1.

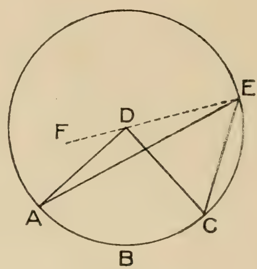


Fig. 2.

It it required to show that $\angle$ ADC = twice $\angle$ AEC.

In both figures :—

Join ED and produce ED to any point F.

In $\triangle$ DAE, DA = DE, $\therefore$ $\angle$ DAE = $\angle$ DEA.

ADF is an exterior $\angle$ of $\triangle$ ADE,

$\therefore$ $\angle$ ADF = $\angle$ DAE + $\angle$ DEA = 2 $\angle$ DEA.

Similarly $\angle$ CDF = 2 $\angle$ DEC.

In Fig. 1 :—

$$\angle \text{ADF} = 2 \angle \text{DEA},$$
$$\angle \text{CDF} = 2 \angle \text{DEC},$$
$$\text{adding,} \quad \angle \text{ADC} = 2 (\angle \text{DEA} + \angle \text{DEC})$$
$$= 2 \angle \text{AEC}.$$

In Fig. 2 :—

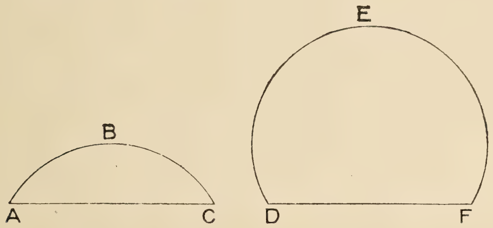
$$\angle CDF = 2 \angle DEC,$$

$$\angle ADF = 2 \angle DEA,$$

subtracting, $\angle ADC = 2 (\angle DEC - \angle DEA)$

$$= 2 \angle AEC.$$

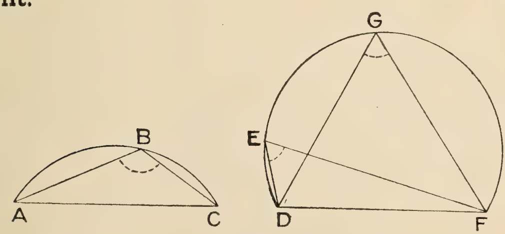
142. **Definition** :—The figure bounded by an arc of a circle and the chord which joins the ends of the arc is called a **segment of a circle.**



ABC, DEF are segments of circles.

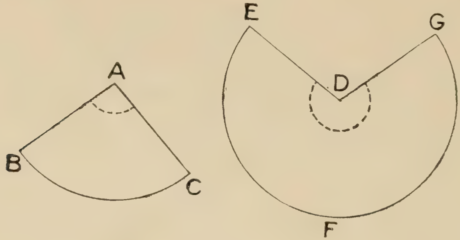
A semi-circle is a particular case of a segment.

143. **Definition** :—If the ends of a chord of a segment be joined to any point in the arc of the segment, the angle between the joining lines is called an **angle in the segment.**



ABC is an $\angle$ **in** the segment ABC, and DEF is an $\angle$ **in** the segment DEF. DGF is also an $\angle$ **in** the segment DEF.

144. **Definition**:—The figure bounded by two radii of a circle, and either of the arcs intercepted by the radii is called a **sector of the circle.**

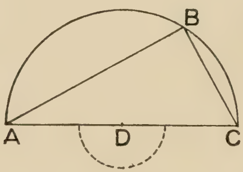


ABC, DEFG are sectors of circles.

BAC is the ∠ of the sector ABC, and the reflex ∠ EDG is the ∠ of the sector DEFG.

145. **Theorem**:—**The angle in a semi-circle is a right angle.**

Let ABC be an ∠ in the semi-circle ABC, of which D is the centre.



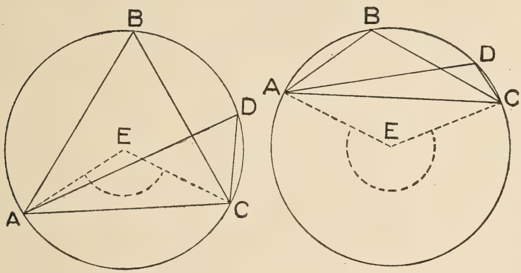
It is required to show that ABC is a rt. ∠.

The ∠ ABC at the circumference, and the st. ∠ ADC at the centre, would each subtend the same arc if the circle were completed.

∴, by § 141, ∠ ABC = $\frac{1}{2}$ ∠ ADC = a rt. ∠.

146. **Theorem:—Angles in the same segment of a circle are equal to each other.**

Let ABC, ADC be two $\angle$s in the same segment ABDC, of which E is the centre.



It is required to show that $\angle$ ABC = $\angle$ ADC.

Join AE, EC.

The $\angle$ AEC at the centre and the $\angle$s ABC and ADC at the circumference are subtended by the same arc,

$$\therefore \angle \text{ABC} = \tfrac{1}{2} \angle \text{AEC}$$

$$\text{and } \angle \text{ADC} = \tfrac{1}{2} \angle \text{AEC},$$

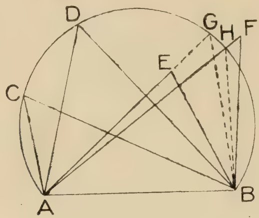
$$\therefore \angle \text{ABC} = \angle \text{ADC}.$$

Alternative statement of the preceding theorem:—

The angle in a given segment is constant in magnitude for all positions of the vertex on the arc.

147. **Theorem :—The locus of all points on one side of a line-segment at which the line-segment subtends equal angles is the arc of a segment of which the line-segment is the chord.**

Let AB be the given line-segment, and C one of the points. Describe a circle through the points ACB.



It is required to show that arc ACB is the locus of all points on the same side of AB at which AB subtends an $\angle$ equal to $\angle$ ACB.

Take any other point D on arc ACB, E any point within the segment, and F any point without the segment.

Join AD, DB, AE, EB, AF, FB.

$\angle$ ADB = $\angle$ ACB. (§ 146.)

Produce AE to meet arc ACB at G. Join BG.

AEB is an exterior $\angle$ of $\triangle$ EGB,

$\therefore$ $\angle$ AEB > $\angle$ AGB;

but $\angle$ AGB = $\angle$ ACB, (§ 146.)

$\therefore$ $\angle$ AEB > $\angle$ ACB.

In a similar manner it may be shown that

$\angle$ AFB < $\angle$ ACB;

and consequently arc ACB is the locus.

148.—Exercises.

1. Show how to find the centre of a circle of which an arc is given.

2. Construct a circular arc on a chord of 3 inches and having the apex 3 inches from the chord. Calculate the radius of the circle.

3. If the chord of an arc is a inches, and the distance of its apex from the chord b inches, show that the radius of the circle is $\frac{a^2 + 4b^2}{8b}$.

4. Two chords AOB, COD, intersect at a point O within the circle. Show that AOC, BOD are similar △s. BOC, AOD are also similar △s. Read the segments that contain the equal ∠s.

5. ABC is a △ inscribed in a circle, and the bisector of ∠A meets the circumference again at D. Show that the st. line drawn from D ⊥ BC is a diameter.

6. A circle is divided into two segments by a chord equal to the radius. Show that the ∠ in the major segment is 30° and that in the minor segment is 150°.

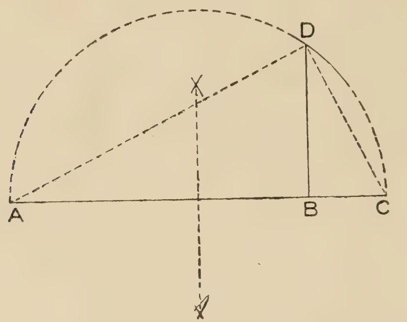
7. **An ∠ in a major segment of a circle is acute; and an ∠ in a minor segment is obtuse.**

8. The locus of the vertices of the rt. ∠s of all rt.-∠d △s on the same hypotenuse is a circle.

Mean Proportionals.

149. **Problem:—To find the mean proportional to two given line-segments.**

Let AB, BC be the two given line-segments, and let them be placed in the same st. line.



It is required to find a mean proportional to AB, BC.

On AC as diameter describe a semi-circle ADC. From B draw BD $\perp$ AC and meeting the arc ADC at D.

BD is the required mean proportional.

Join AD, DC.

By § 145, ADC is a rt. $\angle$, and consequently the $\perp$ DB divides $\triangle$ ADC into two similar $\triangle$s ABD, DBC.

Hence $\frac{AB}{BD} = \frac{DB}{BC}$; that is, DB is a mean proportional between AB, BC.

150.—Exercises.

1. If from any point on the circumference of a circle a $\perp$ be drawn to a diameter, the square on the $\perp$ equals the rect. contained by the segments of the diameter.

2. **Construct a square equal to a given rect.**

3. Construct a square equal to a given ||gm.

4. Construct a square equal to a given $\triangle$.

5. Draw a square having its area 12 sq. inches.

6. AB is a radius of a circle, of which A is the centre. An inner circle is described on AB as diameter. Show that any chord of the outer circle drawn from B is bisected by the inner circle.

7. A is the centre of a circle, and B is any point. Any st. line drawn through B cuts the circumference at C, D. Show that the locus of the middle point of CD is the circumference, or part of the circumference of a circle, of which AB is a diameter.

8. If two circles cut one another, and from one of the points of intersection two diameters be drawn, their extremities and the other point of intersection wlll be in one st. line.

9. Divide a given line-segment into two parts such that the rect. contained by the parts is equal to the square on another given line-segment.

10. If a line-segment be divided into two parts, the rect. contained by the parts is greatest when the line is bisected.

11. AB and C are two given line-segments. Find a point D in AB produced such that rect. AD.DB = sq. on C.

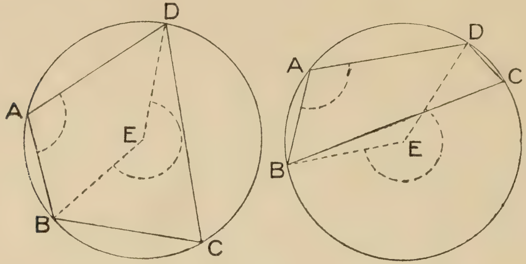
12. Construct a rect. equal in area to a given square and having its perimeter equal to a given line-segment.

When will the solution be impossible?

13. Show how to construct a square equal in area to a given polygon.

151. **Theorem :—If a quadrilateral be inscribed in a circle, its opposite angles are supplementary.**

Let ABCD be a 4-gon inscribed in a circle of which E is the centre.



It is required to show that $\angle$ A + $\angle$ C = 2 rt. $\angle$s.

Join BE, ED.

$\angle$ BED at the centre and $\angle$ C at the circumference are subtended by the same arc BAD,

$\therefore \angle \text{C} = \frac{1}{2} \angle \text{BED}$. (§ 141.)

Similarly $\angle \text{A} = \frac{1}{2}$ reflex $\angle$ BED.

Hence $\angle$ A + $\angle$ C = $\frac{1}{2}$ the sum of the two $\angle$s BED at the centre = $\frac{1}{2}$ of 4 rt. $\angle$s

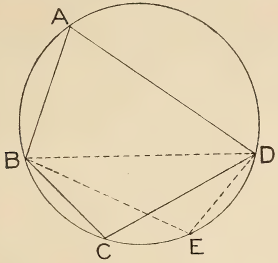
= 2 rt. $\angle$s.

152. **Theorem :—If the opposite angles of a quadrilateral be supplementary, its vertices are concyclic.**

Let ABCD be a 4-gon having $\angle$ A + $\angle$ C = 2 rt. $\angle$s.

It is required to show that A, B, C, D are on the circumference of a circle.

Draw a circle through the three points A, B, D. On this circumference and on the side of BD remote from A take a point E. Join BE, ED.



ABED is a 4-gon inscribed in a circle,

$\therefore \angle A + \angle E = 2$ rt. $\angle$s; (§ 151.)

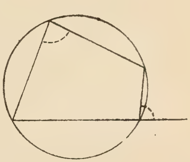
but $\angle A + \angle C = 2$ rt. $\angle$s. (hypothesis.)

$\therefore \angle A + \angle E = \angle A + \angle C$,

and $\angle E = \angle C$.

Consequently, as C, E are on the same side of BD, the circle BADE passes through C. (§ 147.)

153.—Exercises.



1. If one side of an inscribed 4-gon be produced, the exterior $\angle$ thus formed at one vertex equals the interior $\angle$ at the opposite vertex of the 4-gon.

2. From a point O without a circle two st. lines OAB, OCD are drawn cutting the circumference at A, B, C, D. Show that $\triangle$s OBC, OAD are similar and that $\triangle$s OAC, OBD are similar.

3. If a ||gm. be inscribed in a circle, the ||gm. is a rect.

4. A, D, C, E, B are five successive points on the circumference of a circle; and A, B are fixed. Show that the sum of the $\angle$s ADC, CEB is the same for all positions of D, C, E.

5. On a given line-segment as chord construct a segment containing an ∠ equal to a given acute ∠.

6. On a given line-segment as chord construct a segment containing an ∠ equal to a given obtuse ∠.

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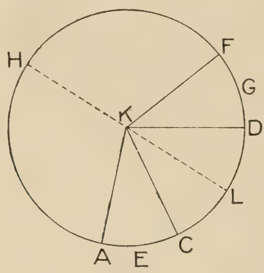
154. **Theorem :—If two angles at the centre of a circle be equal to each other, they are subtended by equal arcs.**

Let AKC, DKF be equal ∠s at the centre K of the circle ACD.



It is required to show that arc AEC equals arc DGF.

Draw the diameter HKL bisecting ∠ CKD.

Suppose the circle to be folded along the diameter HKL, and the semi-circle HFL will coincide throughout with the semi-circle HAL.

∠ LKD = ∠ LKC and ∴ KD falls along KC and consequently D falls on C.

∠ DKF = ∠ CKA and ∴ KF falls along KA, and consequently F falls on A.

Hence the arc DGF coincides with arcs CEA, and consequently these arcs are equal.

155.—Exercises.

1. **If two arcs of a circle be equal to each other, they subtend equal ∠s at the centre.** (Prove either by indirect demonstration, or by the construction and method used in § 154).

2. **If two ∠s at the circumference of a circle be equal to each other, they are subtended by equal arcs.**

3. **If two arcs of a circle be equal to each other, they subtend equal ∠s at the circumference.**

4. In equal circles equal ∠s at the centres (or circumferences) stand on equal arcs.

5. In equal circles equal arcs subtend equal ∠s at the centres (or circumferences).

6. **If two arcs of a circle (or of equal circles) be equal, they are cut off by equal chords.**

7. **If two chords of a circle be equal to each other, the major and minor arcs cut off by one are respectively equal to the major and minor arcs cut off by the other.**

8. If two sectors of a circle have equal ∠s at the centre, the sectors are congruent.

9. **Bisect a given arc of a circle.**

10. Parallel chords of a circle intercept equal arcs.

Show also that the converse is true.

11. If two equal circles cut one another, any st. line drawn through one of the points of intersection will meet the circles again at two points which are equally distant from the other point of intersection.

12. The bisectors of the opposite ∠s of a 4-gon inscribed in a circle meet the circumference at the ends of a diameter.

13. If two ∠ at the centre of a circle be supplementary, the sum of the arcs on which they stand is equal to half the circumference.

14. If any number of ∠s be in a segment, their bisectors all pass through one point.

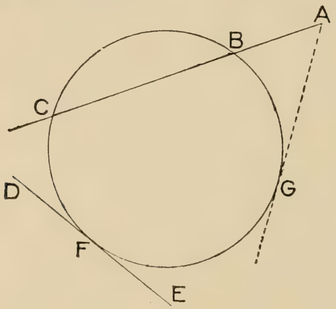
Tangents and Chords.

156. **Definition** :—Any straight line which cuts a circle is called a **secant.**

Definition :—A straight line which, however far it may be produced, has one point on the circumference of a circle, and all other points without the circle is called a **tangent** to the circle.

A tangent is said to **touch** the circle.

Definition :—The common point of a tangent and circle, that is the point where the tangent touches the circle, is called the **point of contact.**



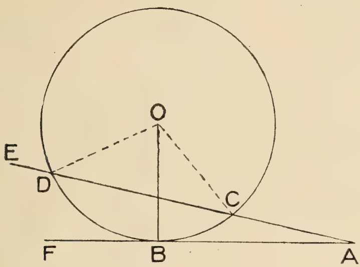
ABC is a secant drawn to the circle BCF from the point A.

DFE is a tangent to the circle BCF, touching the circle at the point of contact F.

If the secant ABC rotate about the point A until the two points B, C where it cuts the circle coincide at G, the secant becomes a tangent having G for the point of contact.

157. **Theorem :—The radius drawn to the point of contact of a tangent is perpendicular to the tangent.**

CBD is a circle and AB a tangent made by the rotation of the secant ACD about the point A. OB is the radius drawn to the point of contact.



It is required to show that OB is $\perp$ to AB.

Join OC, OD.

ODC is an isosceles $\triangle$, and $\therefore \angle$ ODC = $\angle$ OCD.

But st. $\angle$ EDC = st. $\angle$ DCA.

$\therefore \angle$ ODE = $\angle$ OCA.

As OCD rotates about A these $\angle$s are continually equal to each other and finally $\angle$ ODE becomes $\angle$ OBF and $\angle$ OCA becomes $\angle$ OBA.

$\therefore \angle$ OBF = $\angle$ OBA

and consequently OB is $\perp$ to ABF.

158. It follows from the theorem in § 157 that:—if the perpendicular from the centre of a circle to a straight line be a radius of the circle, the straight line is a tangent to the circle with the foot of the perpendicular as the point of contact of the tangent.

159.—Exercises.

1. Draw a tangent to a given circle from a given point on the circumference.

2. Only one tangent can be drawn to a circle from a given point on the circumference.

3. Find the locus of the centres of all circles that touch a given st. line at a given point.

4. Describe a circle to pass through a given point and touch a given st. line at a given point.

5. Tangents at the ends of a diameter are ||.

6. C is any point on the tangent of which A is the point of contact. The st. line from C to the centre O cuts the circumference at B. AD is $\perp$ OC. Show that BA bisects the $\angle$ DAC.

7. Find the locus of the centres of all circles which touch two given || st. lines.

8. Draw a circle to touch two given || st. lines and pass through a given point between the ||s. Show that two such circles may be drawn.

9. Show that any number of circles may be drawn to touch each of two given intersecting st. lines, and that the locus of their centres consists of two st. lines at rt. $\angle$s to each other.

10. Draw a circle to touch two given intersecting st. lines and have its radius equal to a given line-segment. Show that two such circles may be drawn.

11. To a given circle draw two tangents, each of which is || to a given st. line.

12. To a given circle draw two tangents, each of which is $\perp$ to a given st. line.

13. Give an alternative proof for the theorem of § 157 by supposing the radius OB drawn to the point of contact of the tangent ABF not $\perp$ to AF and drawing OG $\perp$ AF.

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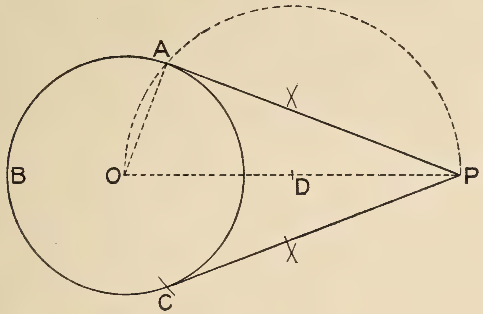
160. **Problem :**—To draw a tangent to a given circle from a given point without the circle.

Let ABC be the given circle, and P the given point.

It is required to draw a tangent from P to the circle ABC.



Join P to the centre O. Bisect OP at D. With centre D and radius DO, describe a circle cutting the circle ABC at A and C. Join PA, PC.

Either PA or PC is a tangent to the given circle.

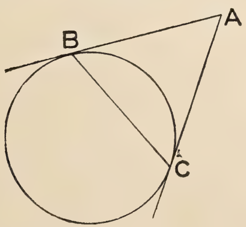
Join OA.

OAP is an ∠ in a semi-circle, and is ∴ a rt.-∠. (§ 145.)

∴ PA is a tangent. (§ 158.)

In the same manner it may be shown that PC is a tangent.

161. **Definition**:—The straight line-segment joining the points of contact of two tangents to a circle is called the **chord of contact** of the tangents.



BC is the chord of contact of the tangents AB, AC.

162.—Exercises.

1. **Tangents drawn to a circle from the same external point are equal to each other.**

2. If from a point without a circle two tangents be drawn, the st. line drawn from this point to the centre bisects the chord of contact and cuts it at rt. $\angle$s.

3. If a 4-gon be circumscribed about a circle, the sum of one pair of opposite sides equals the sum of the other pair.

4. Through a given point draw a st. line, such that the chord intercepted on the line by a given circle is equal to a given line-segment.

5. If a ||gm be circumscribed about a circle, the ||gm is a rhombus.

6. If two tangents to a circle be ||, their chord of contact is a diameter.

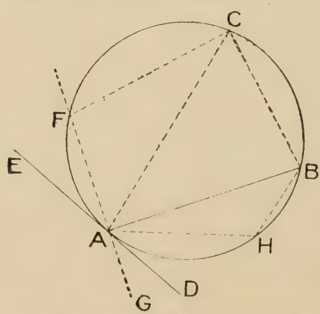
7. If two || tangents to a circle be cut by a third tangent to the circle at A, B; show that AB subtends a rt. $\angle$ at the centre.

8. If a 4-gon be circumscribed about a circle, the $\angle$s subtended at the centre by a pair of opposite sides are supplementary.

9. To a given circle draw two tangents containing an $\angle$ equal to a given $\angle$.

10. Find the locus of the points from which tangents drawn to a given circle are equal to a given line-segment.

163. **Theorem :—If at one end of a chord of a circle a tangent be drawn, each angle between the chord**



and the tangent is equal to the angle in the segment on the other side of the chord.

Let ABC be a circle of which AB is a chord and EAD a tangent.

It is required to show that $\angle$ DAB = $\angle$ ACB, and that $\angle$ EAB = $\angle$ AHB.

In arc AFC take a point F. Join CF and draw the line FAG.

AFCB is a 4-gon inscribed in the circle and $\therefore$ $\angle$ GAB = $\angle$ FCB (§ 153, Ex. 1); and these $\angle$s are equal however near F is to A.

Let F move along the circumference towards A and finally coincide with A.

The line FAG rotates about the point A and coincides with EAD. The $\angle$ GAB becomes $\angle$ DAB and $\angle$ FCB becomes $\angle$ ACB.

$\therefore$ $\angle$ DAB = $\angle$ ACB.

$\angle$ EAB is supplementary to $\angle$ DAB and $\angle$ AHB is supplementary to $\angle$ ACB.

$\therefore$ $\angle$ EAB = $\angle$ AHB.

164.—Exercises.

1. AB is a chord of a circle and AC is a diameter. AD is $\perp$ to the tangent at B. Show that AB bisects the $\angle$ DAC.

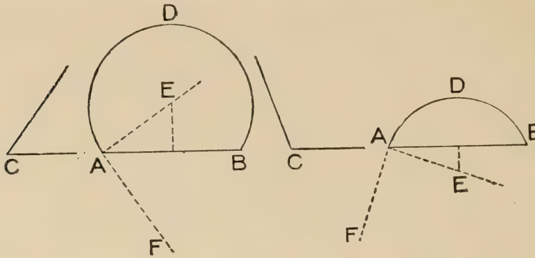
2. Two circles intersect at A and B. Any point P on the circumference of one circle is joined to A and B and the joining lines are produced to meet the circumference of the other circle at C, D. Show that CD is $\parallel$ to the tangent at P.

3. **From a given circle cut off a segment containing an $\angle$ equal to a given $\angle$.**

4. **In a given circle inscribe a $\triangle$ equiangular to a given $\triangle$.**

5. **On a given st. line-segment to make a segment of a circle containing an $\angle$ equal to a given $\angle$.**

Let AB be the line-segment and C the given ∠.



It is required to draw on AB a segment containing an ∠ = ∠ C.

Make ∠ BAF = ∠ C. Draw AE ⊥ AF.

Draw a st. line bisecting AB at rt. ∠s and produce it to cut AE at E.

E is the centre of the required segment.

Show that an ∠ in segment ADB = ∠ C.

6. On a st. line-segment 4 cm. in length describe a segment containing an ∠ of (*a*) 45°, (*b*) 150°, (*c*) 72°, (*d*) 116°.

7. Give an alternative proof for the theorem in § 163 by drawing a diameter AK from the point of contract of the tangent DAE and joining K to B the end of the chord AB.

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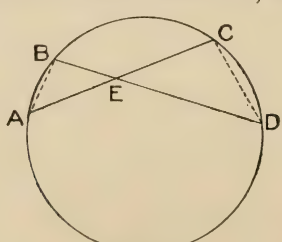
165. **Theorem :—If two chords intersect within a circle, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.**

In the circle ABC let the chords AC, BD intersect at E.



It is required to show that rect. AE.EC = rect. BE.ED.

Join AB, CD.

$\angle$s ABD, ACD are equal to each other, because they are in the same segment ACD.

Similarly $\angle$BAC = $\angle$BDC.

Also $\angle$AEB = $\angle$CED.

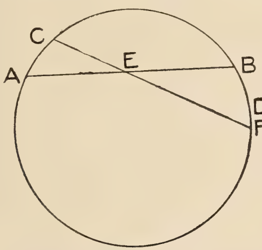
$\therefore$ ABE, DCE are similar $\triangle$s, and consequently

$$\frac{AE}{BE} = \frac{ED}{EC},$$

and $\therefore$ rect. AE.EC = rect. BE.ED.

166. **Theorem**:—**If two straight line-segments cut each other so that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other, the four extremities of the two line-segments are concyclic.**

Let AB, CD cut at E so that rect. AE.EB = rect. CE.ED.



It is required to show that A, C, B and D are concyclic.

Describe a circle through A, C, B, and let it meet ED produced if necessary at F.

Then rect. AE.EB = rect. CE.EF (§ 165).

But, by hypothesis, rect. AE.EB = rect. CE.ED.

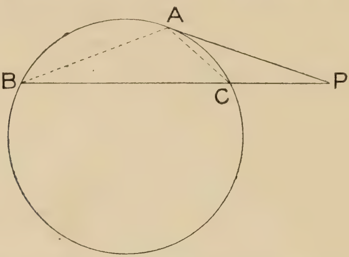
$\therefore$ rect. CE.EF = CE.ED,

and hence EF = ED.

$\therefore$ F coincides with D and the points A, C, B, D are concyclic.

167. **Theorem :—If from a point without a circle, a secant and a tangent be drawn, the square on the tangent is equal to the rectangle contained by the secant, and the part of it without the circle.**

Let PA be a tangent and PCB a secant to the circle ABC.



It is required to show that the sq. on PA = rect. PB.PC.

Join AB, AC.

AP is a tangent, and AC a chord from the same point A

$\therefore$ $\angle$PAC = $\angle$ABC. (§ 163.)

Then in $\triangle$s PAB, PAC,

$\angle$P is common, $\angle$PBA = $\angle$PAC, and $\therefore$ also $\angle$PAB = $\angle$PCA.

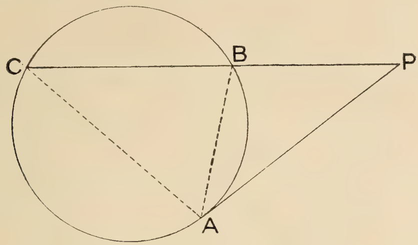
Hence these $\triangle$s are similar,

$$\text{and } \frac{PB}{PA} = \frac{PA}{PC},$$

$\therefore$ sq. on PA = rect. PB.PC.

168. **Theorem :—If from a point without a circle two straight lines be drawn, one of which is a secant and the other meets the circle so that the square on the line which meets the circle is equal to the rectangle contained by the secant and the part of it without the circle, the line which meets the circle is a tangent.**

Let PA and PBC be drawn from P to the circle ABC, so that the sq. on PA = rect. PB.PC.



It is required to prove that PA is a tangent.

Join AB, AC.

In $\triangle$s PAC, PBA, $\angle$ P is common, and since sq. on AP = rect. PC.PB,

$$\frac{PC}{PA} = \frac{PA}{PB},$$

$\therefore$ these $\triangle$s are similar (§ 112); and $\angle$ PAB = $\angle$ PCA.

$\therefore$, by § 163, PA coincides with the tangent at A, that is, PA is a tangent to the circle.

169.—Exercises.

1. PAB, PCD are two secants drawn from a point P without a circle. Show that rect. PA.PB = rect. PC.PD.

2. If in two line-segments PB, PD points A, C respectively be taken such that rect. PA.PB = rect. PC.PD, the four points A, B, C, D are concyclic.

3. If two circles intersect, their common chord bisects their common tangents.

4. If two circles intersect, the tangents drawn to them from any point in their common chord produced are equal to each other.

5. Through P any point in the common chord, or the common chord produced, of two intersecting circles two lines are drawn cutting one circle at A, B, and the other at C, D. Show that A, B, C, D are concyclic.

6. Through a point P within a circle, any chord APB is drawn. If O be the centre, show that rect. AP.PB = $OA^2 - OP^2$.

7. From any point P without a circle any secant PAB is drawn. If O be the centre, show that rect. PA.PB = $OP^2 - OA^2$.

8. From a given point as centre describe a circle cutting a given st. line in two points, so that the rectangle contained by their distances from a given point in the st. line may be equal to a given square.

9. Describe a circle to pass through two given points and touch a given st. line.

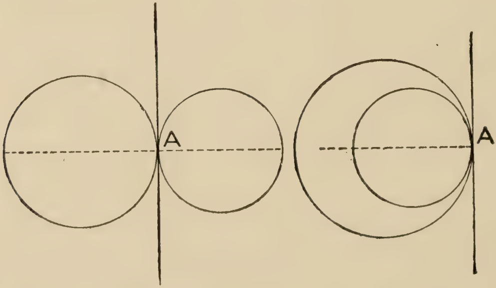
10. If three circles be drawn so that each intersects the other two, the common chords of each pair meet at a point.

11. Find a point D, in the side BC of △ ABC, such that the sq. on AD = rect. BD.DC.

12. Use the theorem in § 167 to find a mean proportional to two given line-segments.

Contact of Circles.

170. **Definition** :—If two circles touch the same straight line at the same point, they are said to **touch** each other at that point.



171. If two circles which touch each other be on opposite sides of the common tangent at their point of contact, and consequently each circle outside the other, they are said

to touch **externally**; if they be on the same side of the common tangent, and consequently one within the other, they are said to touch **internally**.

172. If through the point of contact A, a straight line be drawn perpendicular to the common tangent at A, this straight line passes through the centre of each circle (§ 157); and consequently :—

If two circles touch each other, either internally or externally, the two centres and the point of contact are in the same straight line.

173.—Exercises.

1. If the st. line joining the centres of two circles pass through a point common to the two circumferences, the circles touch each other at that point.

2. Find the locus of the centres of all circles which touch a given circle at a given point.

3. Describe a circle to pass through a given point, and touch a given circle at a given point.

4. If two circles touch each other, any st. line drawn through the point of contact will cut off segments that contain equal $\angle$s.

5. Two circles ACO, BDO touch, and through O, st. lines AOB, COD are drawn. Show that AC $\parallel$ BD.

6. If two $\parallel$ diameters be drawn in two circles which touch one another, the point of contact and an extremity of each diameter are in the same st. line.

7. Describe a circle which shall touch a given circle, have its centre in a given st. line, and pass through a given point in the st. line.

8. Describe three circles having their centres at three given points and touching each other in pairs. Show that there are four solutions.

9. Two circles touch a given st. line at two given points, and also touch each other; find the locus of their point of contact.

10. If through the point of contact of two touching circles a st. line be drawn cutting the circles again at two points, the radii drawn to these points are $\parallel$.

174.—Exercises.

1. ABC is a $\triangle$, AX is $\perp$ BC, and AD is a diameter of the circumscribed circle. Show that rect. AB.AC = rect. AX.AD.

2. P is any point in the side BC produced of the $\triangle$ ABC. Draw the st. line PDE, cutting AB at E, AC at D, and such that rect. PD.PE = rect. PB.PC.

3. If two chords of a circle intersect at rt. $\angle$s, the sum of the squares on their segments is equal to the square on the diameter.

4. Find a point in the circumference of a given circle, the sum of the squares on whose distances from two given points may be a maximum or minimum.

5. AOB, COD are chords cutting at a point O within the circle. Show that $\angle$ BOC equals an $\angle$ at the circumference, subtended by an arc which is equal to the sum of the arcs subtending $\angle$s BOC, AOD.

6. Two chords AB, CD intersect at a point O without a circle. Show that $\angle$AOC equals an $\angle$ at the circumference subtended by an arc which is equal to the difference of the two arcs BD, AC intercepted between OBA and ODC.

7. ABCD is a 4-gon inscribed in a circle and having AC, BD at rt. $\angle$s. From O, the point of intersection of AC, BD a $\perp$ ON is drawn to AB. Show that NO produced bisects CD.

8. ABDC is a 4-gon inscribed in a circle. CD, AB intersect at X; DB, CA at Y. Show that the bisectors of $\angle$s X, Y cut at rt. $\angle$s.

9. Two circles touch externally at E, and are cut by a st. line at A, B, C, D. Show that $\angle$AED is supplementary to $\angle$BEC.

10. Two circles intersect at A. Through A two st. lines are drawn, meeting one circle in D, B, and the other in E, C. Show that DB, EC contain an $\angle$ equal to the $\angle$ between the tangents to the circles at A.

11. If at a point of intersection of two circles the tangents drawn to the circles be at rt. $\angle$s, the st. line joining the points where these tangents meet the circles again, passes through the other point of intersection of the circles.

12. Show that through a point within a circle, which is not the centre, two chords cannot be drawn which shall both be bisected at that point.

13. ABCD is a 4-gon inscribed in a circle; show that the bisectors of $\angle$ A and of the exterior $\angle$ at C meet on the circumference.

14. Find a point within a given $\triangle$ at which the three sides subtend equal $\angle$s. When is the solution possible?

15. Construct a $\triangle$ having given the base, the vertical $\angle$, and

(*a*) the median drawn to the base;
(*b*) the altitude;
(*c*) the sum of the other two sides;
(*d*) the difference of the other two sides.

16. Through one of the points of intersection of two given circles draw the greatest possible line-segment terminated in the two circumferences.

17. Through one of the points of intersection of two given circles draw a line-segment terminated in the two circumferences and equal to a given line-segment.

18. Describe a circle of given radius to touch two given circles.

19. DEF is a st. line cutting BC, CA, AB, the sides of $\triangle$ ABC, at D, E, F respectively. Show that the circles circumscribed about the $\triangle$s AEF, BFD, CDE, ABC, all pass through one point.

20. Two circles touch at A and BAC is drawn terminated in the circumferences at B, C. Show that the tangents at B, C are $\parallel$.

21. If two circles cut each other, any two $\parallel$ line-segments drawn through the points of intersection of the circles and terminated in the circumferences are equal.

22. D, E, F are any points on the sides BC, CA, AB of $\triangle$ ABC. Show that the circles circumscribed about the $\triangle$s AFE, BDF, CED pass through a common point.

23. Two arcs stand on a common chord AB. P is any point on one arc and PA, PB cut the other arc at C, D. Show that the length of CD is constant.

24. ACB is an $\angle$ in a segment. The tangent at A is $\parallel$ to the bisector of $\angle$ ACB and meets BC produced at D. Show that AD = AB.

CHAPTER VI.

Orthocentre of a Triangle.

175. **Theorem :—The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.**

Let ABC be a △ having BY, CZ ⊥ CA, AB respectively and meeting at O.

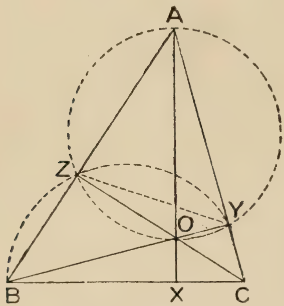


Fig. 1.

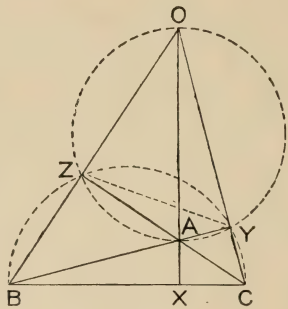


Fig. 2.

It is required to show that AO produced is ⊥ BC ; that is, to show that the ⊥s from A, B, C to BC, CA, AB respectively meet at a common point O

Join YZ.

In (Fig. 1)

∠ BYC = ∠ BZC, ∴ points B, Z, Y, C are concyclic (§ 147); and hence ∠ BCZ = ∠ BYZ. (§ 146.)

∠ AZO + ∠ AYO = 2 rt. ∠s, ∴ points O, Z, A, Y are concyclic (§ 152) ; and hence ∠ OYZ = ∠ OAZ. ∴ ∠ BCZ = ∠ OAZ.

Then in $\triangle$s AOZ, COX, $\angle$XCO = $\angle$OAZ, and $\angle$XOC = $\angle$ZOA,

$$\therefore \angle OXC = \angle OZA = \text{a rt. } \angle.$$

A similar proof may be given for Fig. 2.

The point O, where the $\perp$s intersect, is the orthocentre of the $\triangle$.

176.—Exercises.

1. In $\triangle$ ABC, AX, BY, CZ the $\perp$s to BC, CA, AB intersect at O. Show that :—

(*a*) rect. AO.OX = rect. BO.OY = rect. CO.OZ ;

(*b*) rect. AB.AZ = rect. AO.AX = rect. AC.AY ;

(*c*) if AX meet the circumscribed circle of $\triangle$ ABC at K, OX = XK ;

(*d*) if S be the centre of the circumscribed circle of $\triangle$ ABC and SD be $\perp$ BC, AO = twice SD ;

(*e*) $\triangle$s AYZ, BZX, CXY, ABC are similar ;

(*f*) AX, BY, CZ bisect the $\angle$s of the pedal $\triangle$ XYZ ; (Note : XYZ is called the *pedal*, or *orthocentric*, $\triangle$ of the $\triangle$ ABC) ;

(*g*) of the four points A, B, C, O, each is the orthocentre of the $\triangle$ of which the other three points are the vertices ;

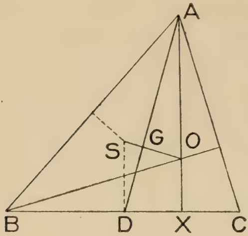
(*h*) If a $\triangle$ LMN be formed by drawing through A, B, C lines MN, NL, LM $\parallel$ BC, CA, AB respectively, O is the circumscribed centre of $\triangle$ LMN ;

(*i*) If S be the centre of the circumscribed circle of $\triangle$ ABC, AS, BS, CS are respectively $\perp$ YZ, ZX, XY the sides of the orthocentric $\triangle$;

(*j*) the product of the numerical measures of BX, CY, AZ is equal to the product of the numerical measures of XC, YA, ZB. (See § 101, Ex. 13).

2. Given the base and vertical $\angle$ of a $\triangle$, find the locus of its orthocentre.

3. If the base BC and vertical ∠A of a △ ABC be given, and the base be trisected at D, E, the locus of the centroid is an arc containing an ∠ equal to ∠ A, and having DE as its chord.



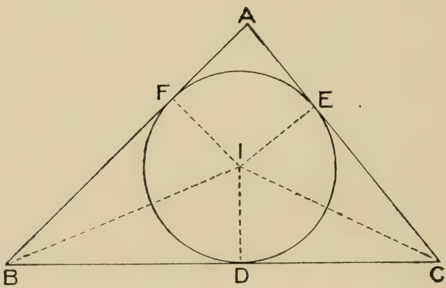
4. ABC is a △ having AX ⊥ BC, AD a median, O the orthocentre, and S the circumscribed centre. Show that OS cuts AD at the centroid G. (Use Ex. 1, *d.*) Show also that G is a point of trisection in SO.

A general enunciation of these results may be given as follows:—**The Orthocentre, Centroid, and the centre of the Circumscribed Circle of a △ are in the same st. line, and the Centroid is a point of trisection in the line-segment joining the other two.**

Inscribed and Escribed Circles.

177. **Problem:—To inscribe a circle in a given triangle.**

ABC is a △.



It is required to inscribe a circle in △ ABC.

Bisect ∠s B, C and let the bisectors meet at I. From I draw ⊥s ID, IE, IF to BC, CA, AB respectively.

In △s BIF, BID,

∠ IBF = ∠ IBD, ∠ IFB = ∠ IDB and BI is common,

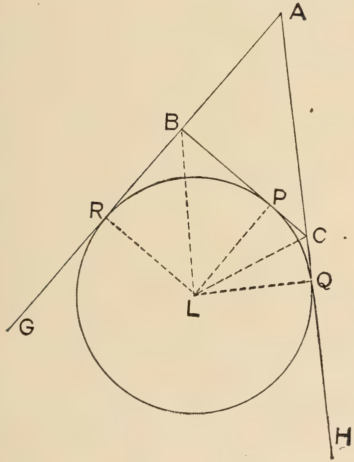
∴ DI = FI.

Similarly IE = ID.

And as the $\angle$s at D, E, F are rt. $\angle$s, a circle described with centre I and radius ID will pass through E, F and touch the sides of $\triangle$ ABC at D, E, F.

178. **Problem:—To draw an escribed circle of a given triangle.**

Let ABC be a given $\triangle$ having AB, AC produced to G, H.



It is required to describe a circle touching the side BC and the two sides AB, AC produced.

Bisect $\angle$s GBC, HCB and let the bisectors meet at L. Draw $\perp$s LP, LQ, LR to BC, CH, BG respectively.

In $\triangle$s LBP, LBR,

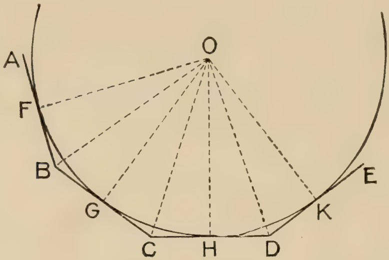
$\angle$LBP = $\angle$LBR, $\angle$LPB = $\angle$LRB and LB is common,

$\therefore$ LP = LR.

Similarly LQ = LP, and as $\angle$s at P, Q, R are rt. $\angle$s, a circle described with centre L and radius LP will pass through Q and R and touch BC, CH, BG at PQR respectively.

179. **Problem:—To inscribe a circle in a given regular polygon.**

Let AB, BC, CD, DE be four consecutive sides of a given regular polygon.



It is required to inscribe a circle in the polygon.

Bisect $\angle$s BCD, CDE and produce the bisectors to meet at O. Join OB. From O draw $\perp$s OF, OG, OH, OK to AB, BC, CD, DE respectively.

In $\triangle$s OCB, OCD,

BC = CD, CO is common, and $\angle$ OCB = $\angle$ OCD,

$\therefore$ $\angle$ OBC = $\angle$ ODC.

But $\angle$ ODC = $\frac{1}{2}$ $\angle$ CDE and $\angle$ ABC = $\angle$ CDE,

$\therefore$ $\angle$ OBC = $\frac{1}{2}$ $\angle$ ABC.

In the same manner it may be shown that if O be joined to all the vertices of the polygon the joining lines will bisect the $\angle$s at the vertices.

In $\triangle$s OCG, OCH,

$\angle$OCG = $\angle$OCH, $\angle$OGC = $\angle$OHC and OC is common,

$\therefore$ OG = OH.

In the same manner it may be shown that the $\perp$s from O to all of the sides are equal to each other, and as the $\angle$s at F, G, H, etc., are rt. $\angle$s, a circle described with O as centre and OF as radius will touch each of the sides and be inscribed in the polygon.

180.—Exercises.

1. Find the locus of the centres of all circles which touch two given intersecting st. lines.

2. The bisectors of the ∠s of a △ are concurrent.

3. The bisectors of the exterior ∠s at two vertices of a △ and the bisector of the interior ∠ at the third vertex are concurrent.

4. If a, b, c be the numerical measures of the sides of △ ABC, and $s = \frac{1}{2}(a + b + c)$,

   (a) in the diagram of § 177, AF $= s - a$, BD $= s - b$, CE $= s - c$;

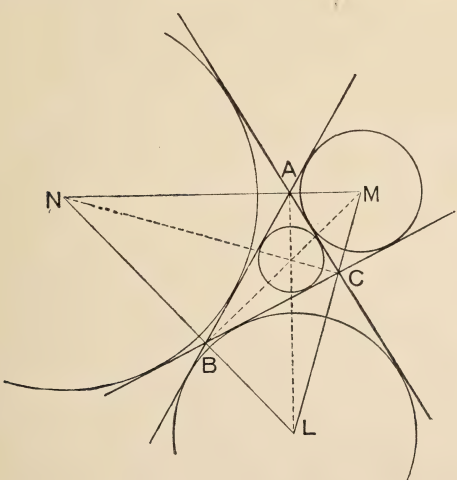
   (b) in the diagram of § 178, AR $= s$, BP $= s - c$, CP $= s - b$;

   (c) if r be the radius of the inscribed circle, rs = area of △ ABC;

   (d) if r_1 be the radius of the escribed circle touching BC, $r_1(s - a)$ = area of △ ABC.

5. If the base and vertical ∠ of a △ be given, find the locus of the inscribed centre.

6. If the base and vertical ∠ of a △ be given, find the loci of the escribed centres.



7. If ABC be a △, L, M, N the centres of the escribed circles, the centre of the inscribed circle is the orthocentre and ABC the pedal △ of the △ LMN.

8. Show that ∠ LMN in figure of Ex. 7 equals half the sum of ∠s BAC, BCA.

9. If the centres of the escribed circles be joined, and the points of contact of the inscribed circle be joined, the △s thus formed are similar.

10. Construct a △ having given the base, the vertical ∠ and the radius of the inscribed circle.

11. Find the centre of a circle cutting off three equal chords of given length from the sides of a △.

12. An escribed circle of △ ABC touches BC at D and also touches AB and AC produced. The inscribed circle touches BC at E. Show that DE equals the difference of AB and AC.

13. Construct a △ having given the centres of the inscribed circle and of two of the escribed circles.

14. Draw circles to touch three st. lines, two of which are ||.

15. Circumscribe a circle about a given regular polygon.

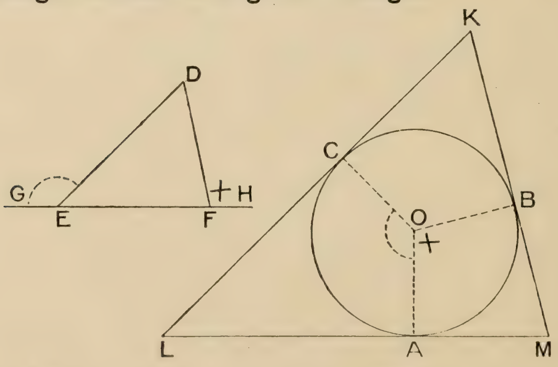
16. Inscribe a square in a given circle.

17. Circumscribe a square about a given circle.

18. Inscribe a circle in a given square.

19. Circumscribe a circle about a given square.

181. **Problem :—About a given circle to circumscribe a triangle similar to a given triangle.**



Let ABC be the given circle, and DEF the given △.

About the circle ABC it is required to circumscribe a $\triangle$ similar to $\triangle$ DEF.

Produce EF to G and H. Draw any radius OA of the circle ABC, and at O make $\angle$s AOB, AOC respectively equal to $\angle$s DFH, DEG, and produce the arms to cut the circle at B, C. At A, B, C draw tangents to the circle meeting at K, L, M.

K, L, M is the required $\triangle$.

AOBM is a 4-gon containing two rt. $\angle$s. OAM, OBM,
$\therefore \angle$ AMB + $\angle$ AOB = 2 rt. $\angle$s. = $\angle$ DFH + $\angle$ DFE.

But $\angle$ AOB = $\angle$ DFH;

and $\therefore$ $\angle$ M = $\angle$ DFE.

In the same manner it may be shown that $\angle$ L = $\angle$ DEF; and consequently $\angle$ K = $\angle$ D.

182.—Exercises.

1. About a given circle circumscribe an equilateral $\triangle$.

2. If two $\triangle$s equiangular to each other be circumscribed about the same circle, the $\triangle$s are congruent.

3. If two $\triangle$s equiangular to each other be inscribed in the same circle, the $\triangle$s are congruent.

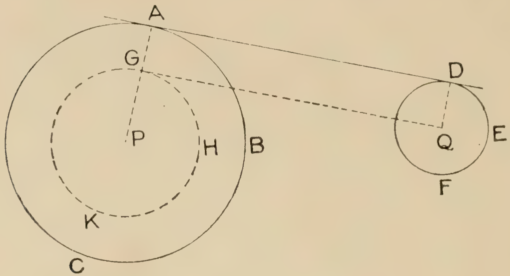
4. Describe a $\triangle$ ABC similar to a given $\triangle$ and such that a given circle is touched by BC and by AB and AC produced.

Analysis of a Problem—Common Tangents of Circles.

183. A common method of discovering the solution of a problem begins with the drawing of the given figure or figures. The required part is then sketched in, and a careful examination is made to determine the connection between the given parts and the required result. Properties of the figure are noted, and lines are drawn that may help in finding the solution. This method of attack is known as the **Analysis of the Problem.** Its use is illustrated in the following sections.

184. **Problem :—To draw the direct common tangents to two given circles.**

Let ABC, DEF be two circles, with centres P, Q.



It is required to draw a direct common tangent to the circles ABC, DEF.

Suppose AD to be a direct common tangent touching the circles at A, D.

Join PA, QD.

PA, QD are both $\perp$ AD, and $\therefore$ PA $\|$ QD.

Cut off AG = DQ. Join QG.

AG is both = and $\|$ QD, $\therefore$ AQ is a $\|$gm, and as $\angle$ GAD is a rt. $\angle$ AQ is a rect.

Draw a circle with centre P and radius PG.

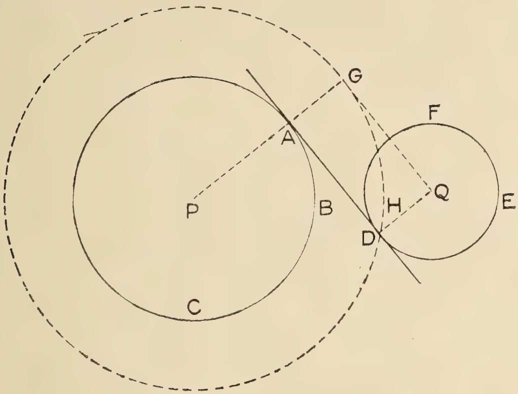
PGQ is a rt. $\angle$, $\therefore$ QG is a tangent to the circle GHK and this tangent is drawn from the given point Q. The radius PG of the circle GHK is the difference of the radii of the given circles.

Using the construction suggested by the above analysis the pupil should make the direct drawing and prove that it is correct.

Show that two direct common tangents may be drawn.

185. **Problem:—To draw the transverse common tangents to two given circles.**

Let ABC, DEF be two circles with centres P, Q.



It is required to draw a transverse common tangent to the circles ABC, DEF.

Suppose AD to be a transverse common tangent touching the circles at A, D.

Join PA, QD.

PA, QD are both $\perp$ AD, $\therefore$ PA $\parallel$ QD.

Produce PA to G making AG = DQ. Join QG.

Then AQ is seen to be a rect., and if a circle be drawn with centre P and radius PG, QG is seen to be a tangent to this circle. The radius PG of the circle GHK is the sum of the radii of the given circles.

From this analysis the pupil can make the direct construction and give the proof.

Two transverse common tangents may be drawn to the given circles.

186.—Exercises.

1. Draw diagrams to show that the number of common tangents to two circles may be 4, 3, 2, 1 or 0.

2. Draw a st. line to cut two given circles so that the chords intercepted on the line may be equal respectively to two given line-segments.

3. P, Q are the centres of two circles. A common tangent (either direct or transverse) meets the line of centres at R. Show that the ratio PR : QR equals the ratio of the radii of the circles.

4. The transverse common tangents and the line of centres of two circles are concurrent.

5. The direct common tangents and the line of centres of two circles are concurrent.

6. P, Q are the centres of two circles and PA, QB any two ∥ radii drawn in the same direction from P, Q. Show that AB produced and the direct common tangents meet the line of centres at the same point.

7. P, Q are the centres of two circles and PA, QB any two ∥ radii drawn in opposite directions from P, Q. Show that AB and the transverse common tangents meet the line of centres at the same point.

The Nine-Point Circle.

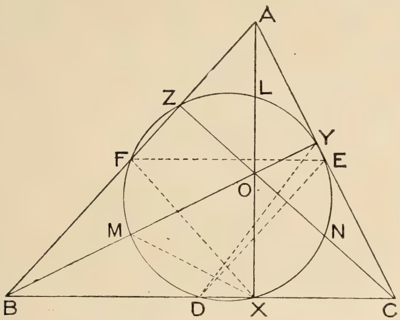
187. **Theorem :—The three middle points of the sides of a triangle, the three projections of the vertices on the opposite sides, and the three middle points of the straight line-segments joining the vertices to the orthocentre are all concyclic.**

Let ABC be a △, AX, BY, CZ the ⊥s from A, B, C to BC, CA, AB respectively, O the orthocentre, L, M, N the middle points of AO, BO, CO respectively, D, E, F the middle points of BC, CA, AB respectively.

It is required to show that the nine points X, Y, Z, L, M, N, D, E, F are concyclic.

Join DE, EF, FX.

D, E, F are middle points of sides of $\triangle$ ABC, $\therefore$ DEFB is a ||gm, and $\angle$DEF $=$ $\angle$DBF.



ABX is a rt.-$\angle$d $\triangle$, F the middle point of the hypotenuse; $\therefore$ FX $=$ FB, and $\angle$FXB $=$ $\angle$FBX.

$\therefore$ $\angle$FXB $=$ $\angle$DEF, and consequently the circle which passes through D, E, F passes also through X.

Similarly it may be shown that this circle passes through Y and Z.

Join MX, YD.

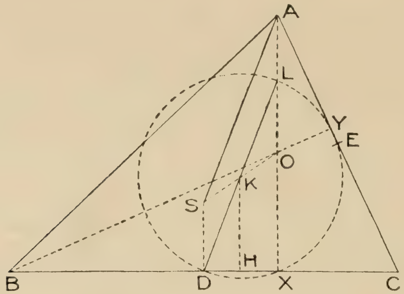
BYC is a rt.-$\angle$d $\triangle$, D the middle point of the hypotenuse; $\therefore$ DY $=$ DB, and $\angle$YDC $=$ $\angle$DBY $+$ $\angle$DYB $=$ twice $\angle$DBY.

In same way in rt.-$\angle$d $\triangle$ $\angle$OBX, $\angle$OMX $=$ twice $\angle$OBD, $\therefore$ $\angle$YMX $=$ $\angle$YDX, and consequently the circle which passes through X, Y, D also passes through M.

Similarly it may be shown that this circle passes through L and N.

$\therefore$ the nine points are concyclic.

188. **Theorems:—The centre of the N.-P. circle is at the middle point of the straight line joining the orthocentre to the circumcentre; and the diameter of the N.-P. circle equals the radius of the circumscribed circle.**



S, O are the circumcentre and orthocentre respectively of △ ABC.

DX is a chord of the N.-P. circle, hence if DX be bisected at H and a ⊥ to DX erected at H this ⊥ must pass through the centre.

Similarly a st. line bisecting YE at rt. ∠s will pass through the centre.

But the ⊥s bisecting DX, EY both bisect SO (§ 103, Ex. 10).

∴ K the middle point of SO is the centre.

(See Example 4, § 176: The centroid is at a point of trisection in this same line-segment SO.)

Join DL and SA.

DL is the diameter of the N.-P. circle since DXL is a tr.-∠, and SA is the radius of the circumscribed circle.

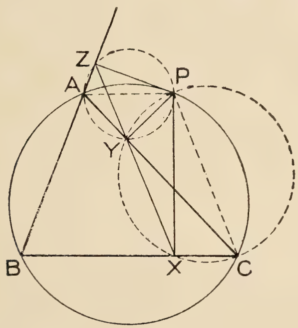
AL, SD are both ⊥ BC, ∴ AL ∥ SD; and SD = AL (§ 176, Ex. 1, *d*).

Hence LD = AS.

SIMPSON'S LINE.

189. **Theorem :—If any point be taken on the circumference of the circumscribed circle of a triangle, the projections of this point on the three sides of the triangle are collinear.**

Let P be any point on the circle ABC, X, Y, Z, the projections of P on BC, CA, AB respectively.



It is required to show that X, Y, Z are in the same st. line.

Join ZY, YX, PC, PA.

$\angle$PYC = $\angle$PXC, $\therefore$ P, Y, X, C are concyclic, and $\angle$XYC = $\angle$XPC.

$\angle$AZP + $\angle$AYP = 2 rt. $\angle$s, $\therefore$ A, Z, P, Y are concyclic, and $\angle$AYZ = $\angle$APZ.

APCB is a cyclic 4-gon, $\therefore$ $\angle$APC + $\angle$B = 2 rt. $\angle$s.

In 4-gon BZPX, $\angle$s BZP, BXP are rt. $\angle$s, $\therefore$ $\angle$ZPX + $\angle$B = 2 rt. $\angle$s.

Hence $\angle$ZPX = $\angle$APC, and as the part APX is common to these $\angle$s,

$\angle$APZ = XPC.

$\therefore$ $\angle$AYZ = $\angle$XYC, and

$\angle$XYC + $\angle$CYZ = $\angle$AYZ + CYZ = 2 rt. $\angle$s;

$\therefore$ XY and YZ are in the same st. line.

190.—Exercises.

1. Draw four circles each of radius $1\frac{3}{4}$ inches, touching a fixed circle of radius 1 inch and also touching a st. line $1\frac{1}{2}$ inches distant from the centre of the circle.

2. Inscribe a circle in a given sector.

3. If two circles touch externally at A and are touched at B, C by a st. line, the line-segment BC subtends a rt. ∠ at A.

4. The circles described on two sides of a △ as diameters intersect on the third side.

5. Of all △s of given base and vertical ∠, the isosceles △ has the greatest area.

6. Each of two equal circles passes through the centre of the other, AB is their common chord. Through **A** a st. line is drawn cutting the circles at C, **D.** Show that △ BCD is equilateral.

7. ABC is an equilateral △ inscribed in a circle, P is any point on the circumference. Of the three line-segments PA, **PB,** PC, show that one equals the sum of the other two.

8. Construct a rt.-∠d △, given the radius of the inscribed circle and an acute ∠ of the △.

9. The diagonals AC, BD of a cyclic 4-gon ABCD cut at E. Show that the tangent at E to the circle circumscribed about △ ABE is ‖ to CD.

10. A, B, C are three points on a circle. The bisector of ∠ ABC meets the circle again at D. DE is drawn ‖ to AB and meets the circle again at E. Show that DE = BC.

11. The side of an equilateral △ circumscribed about a circle is double the side of the equilateral △ inscribed in the same circle.

12. AB is the diameter of a circle and CD a chord. EF is the projection of AB on CD. Show that CE = DF.

13. Construct an isosceles △, given the base and the radius of the inscribed circle.

14. Two circles touch externally. Find the locus of the points from which tangents drawn to the circles are equal to each other.

15. Two circles, centres C, D, intersect at A, B. PAQ is a st. line cutting the circles at P, Q. PC, QD intersect at R. Find the locus of R.

16. Two circles touch internally at A; BC, a chord of the outer circle, touches the inner circle at D. Show that AD bisects ∠BAC.

17. P is a given point on the circumference of a circle, of which AB is a given chord. Through P draw a chord PQ that is bisected by AB.

18. On a given base construct a △ having given the vertical ∠ and the ratio of the two sides.

19. AB is a given line-segment and P, Q are two points such that AP : PB = AQ : QB. Show that the bisectors of ∠s APB, AQB cut AB at the same point.

20. AB is a given line-segment and P, Q are two points such that AP : PB = AQ : QB. Show that the bisectors of the exterior ∠s at P, Q of the △s APB, AQB meet AB produced at the same point.

21. AB is a given line-segment and P is a point which moves so that the ratio AP : PB is constant. The bisectors of the interior and exterior ∠s at P of the △ APB, meet AB and AB produced at C, D respectively. Show that the locus of P is a circle on CD as diameter.

22. AB is a line-segment 2 inches in length. P is a point such that AP is twice BP. Construct the locus of P.

23. Two circles touch externally, and A, B are the points of contact of a common tangent. Show that AB is a mean proportional between their diameters.

24. ABC is a △; L, M, N the centres of its escribed circles. Show that the circle circumscribed about ABC is the N.-P. circle of △ LMN.

25. If on equal chords segments of circles be described containing equal ∠s, the circles are equal.

26. The three circles which go through two vertices of a △ and its orthocentre are each equal to the circle circumscribed about the △.

27. The ⊥ from the middle point of a side of a △ on the opposite side of the pedal △ bisects that side.

28. The locus of the middle points of all st. lines drawn from a fixed point to the circumference of a given circle is a circle.

29. Given the base and vertical ∠ of a △ find the locus of the centre of its N.-P. circle.

30. Draw a circle to pass through a given point and touch two given st. lines.

31. Draw a circle to touch a given circle and two given st. lines.

32. Draw a circle to pass through two given points and touch a given circle.

33. Construct a rt.-∠d △ given the hypotenuse and the radius of the inscribed circle.

34. In △ ABC the inscribed circle touches AB, AC at D, E respectively. The line joining A to the centre cuts the circle at F. Show that F is the centre of the inscribed circle of △ ADE.

35. The inscribed circle of the rt.-∠d △ ABC touches the hypotenuse BC at D. Show that rect. BD, DC = △ ABC.

36. If on the sides of any △ equilateral △s be described outwardly, the centres of the circumscribed △s of the three equilateral △s are the vertices of an equilateral △.

37. Describe three circles to touch each other externally and a given circle internally.

38. Show that two circles can be described with the middle point of the hypotenuse of a rt.-∠d △ as centre to touch the two circles described on the two sides as diameters.

39. A line-segment AB of fixed length moves so as to be constantly ‖ to a given st. line and A to be on the circumference of a given circle. Show that the locus of B is an equal circle.

40. Construct a △ given a vertex, the circumscribed circle and the orthocentre.

41. Construct an isosceles △ equal in area to a given △ and having the vertical ∠ equal to one of the ∠s of the given △.

42. If two chords AB, AC, drawn from a point A in the circumference of the circle ABC, be produced to meet the tangent at the other extremity of the diameter through A in D, E respectively, then the △ AED is similar to △ ABC.

43. If a st. line-segment be divided into two parts, the sq. on the line-segment equals the sum of the rectangles contained by the line-segment and the two parts.

44. ABCD is a 4-gon inscribed in a circle. AB, DC meet at E and BC, AD meet at F. Show that the sq. on EF equals the sum of the sqs. on the tangents drawn from E, F to the circle.

45. The line-segment AB is divided at C so that AC = 3 CB. Circles are described on AC, CB as diameters and a common tangent meets AB produced at D. Show that BD equals the radius of the smaller circle.

46. DE is a diameter of a circle and A is any point on the circumference. The tangent at A meets the tangents at D, E at B, C respectively. BE, CD meet at F. Show that AF is $\parallel$ to BD.

47. TA, TB are tangents to a circle of which C is the centre. AD is $\perp$ BC. Show that TB : BC = BD : DA.

48. ABCD is a 4-gon inscribed in a circle. BA, CD produced meet at P, and AD, BC produced meet at Q. Show that PC : PB = QA : QB.

49. Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in the ratio of two given st. line-segments.

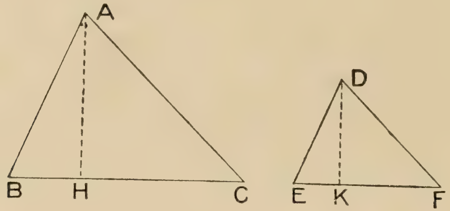
50. Describe a circle to pass through a given point and touch a given st. line and a given circle.

CHAPTER VII.

Areas of Similar Figures.

191. **Theorem:—The areas of similar triangles are proportional to the squares on corresponding sides.**

Let ABC, DEF be similar $\triangle$s, of which BC, EF are corresponding sides.



It is required to show that

$$\frac{\triangle \text{ ABC}}{\triangle \text{ DEF}} = \frac{\text{sq. on BC.}}{\text{sq. on EF.}}$$

Draw AH, DK $\perp$ BC, EF respectively.

Let a, b, c, d be the numerical measures of BC, EF, AH, DK respectively.

$\triangle$s AHC, DKF are similar, $\therefore \dfrac{\text{AH}}{\text{DK}} = \dfrac{\text{AC}}{\text{DF}}$.

$\triangle$s ABC, DEF are similar, $\therefore \dfrac{\text{AC}}{\text{DF}} = \dfrac{\text{BC}}{\text{EF}}$.

Hence $\dfrac{\text{AH}}{\text{DK}} = \dfrac{\text{BC}}{\text{EF}}$, and consequently $\dfrac{c}{d} = \dfrac{a}{b}$.

$\triangle$ ABC $= \frac{1}{2}\, ac$, and $\triangle$ DEF $= \frac{1}{2}\, bd$.

$$\therefore \frac{\triangle \text{ ABC}}{\triangle \text{ DEF}} = \frac{\frac{1}{2}\, ac}{\frac{1}{2}\, bd} = \frac{a}{b} \times \frac{c}{d} = \frac{a^2}{b^2} = \frac{\text{sq. on BC}}{\text{sq. on EF}}.$$

192.—Exercises.

1. Two similar $\triangle$s have corresponding sides in the ratio of 3 to 5. What is the ratio of their areas?

2. The ratio of the areas of two similar $\triangle$s equals the ratio of 64 to 169. What is the ratio of their corresponding sides?

3. Draw a $\triangle$ having sides 4 cm., 5 cm., 6 cm. Make a second $\triangle$ having its area four times that of the first, and divide it into parts each equal and similar to the first $\triangle$.

4. Show that the areas of similar $\triangle$s are as :—

   (*a*) the squares on corresponding altitudes;

   (*b*) the squares on corresponding medians;

   (*c*) the squares on the bisectors of corresponding $\angle$s.

5. ABC, DEF are two similar $\triangle$s such that area of $\triangle$ DEF is twice that of $\triangle$ ABC. What is the ratio of corresponding sides?

Draw $\triangle$ ABC having sides 5 cm., 6 cm., 7 cm., and make $\triangle$ DEF similar to $\triangle$ ABC, and of double the area.

6. If ABC, DEF be similar $\triangle$s of which BC, EF are corresponding sides, and the line-segment G be such that BC : EF = EF : G, then $\triangle$ ABC : $\triangle$ DEF = BC : G; that is :—

If three line-segments be in continued proportion, the first is to the third as any $\triangle$ on the first is to the similar $\triangle$ similarly described on the second.

Note :—Similar $\triangle$s are said to be similarly described on corresponding sides.

7. ABC is a $\triangle$ and G is a line-segment. Describe a $\triangle$ DEF similar to $\triangle$ ABC and such that $\triangle$ ABC : $\triangle$ DEF = BC : G.

Describe another $\triangle$ HKL similar to ABC and such that $\triangle$ ABC : $\triangle$ HKL = AB : G.

8. Bisect a given $\triangle$ by a st. line drawn $\parallel$ to one of its sides.

9. From a given $\triangle$ cut off a part equal to one-third of its area by a st. line drawn $\parallel$ to one of its sides.

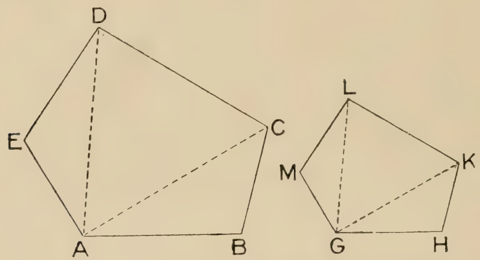
10. Trisect a given $\triangle$ by st. lines drawn $\parallel$ to one of its sides.

11. Show that the equilateral $\triangle$ described on the hypotenuse of a rt.-$\angle$d $\triangle$ equals the sum of the equilateral $\triangle$s on the two sides.

193. **Definition**:—If two polygons of the same number of sides have the angles of one taken in order around the figure respectively equal to the angles of the other in order and have also the corresponding sides in proportion, the polygons are said to be **similar polygons.**

194. **Problem:—On a given straight line-segment to make a polygon similar to a given polygon.**

Let ABCDE be the given polygon and GH the given line-segment.



It is required to describe on GH a polygon similar to ABCDE and such that AB and GH are corresponding sides.

Join AC, AD.

Make $\angle$ H = $\angle$ B, $\angle$ HGK = $\angle$ BAC and produce the arms to meet at K. Make $\angle$ KGL = $\angle$ CAD, $\angle$ GKL = $\angle$ ACD, and produce the arms to meet at L. Make $\angle$ LGM = $\angle$ DAE, $\angle$ GLM = $\angle$ ADE and produce the arms to meet at M.

G H K L M is the required polygon.

$\angle H = \angle B$, $\angle HGK = \angle BAC$, $\therefore$ $\angle HKG = \angle BCA$.

Similarly $\angle GLK = \angle ADC$, and $\angle M = \angle E$.

Hence $\angle HKL = \angle BCD$, $\angle KLM = \angle CDE$ and $\angle HGM = \angle BAE$.

$\therefore$ polygon GHKLM has its $\angle$s equal respectively to the $\angle$s of polygon ABCDE.

From the similar $\triangle$s GHK, ABC, $\frac{HK}{BC} = \frac{GH}{AB} = \frac{KG}{CA}$;

and from the similar $\triangle$s GKL, ACD, $\frac{KG}{CA} = \frac{KL}{CD}$;

$$\therefore \frac{GH}{AB} = \frac{HK}{BC} = \frac{KL}{CD}.$$

In the same manner it may be shown that each of these ratios equals $\frac{LM}{DE}$ and $\therefore$ equals $\frac{MG}{EA}$.

Hence the corresponding sides of the two polygons are proportional; and polygon GHKLM is similar to polygon ABCDE.

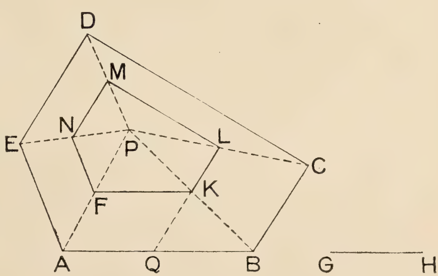
195.—Exercises

1. ABCD is a 4-gon. Construct a 4-gon similar to ABCD and having each side one-third of the corresponding side of ABCD.

2. ABCD is a given 4-gon, E, F two given st. line-segments. Construct a similar 4-gon such that each side has to the corresponding side of ABCD the ratio of E to F.

3. ABCDE is a given polygon and GH a given line-segment. Cut off AQ = GH. Take any point P within ABCDE. Join P to A, B, C, D, E. Draw QK || AP, KF || AB, FN || AE, NM || ED, KL || BC. Join LM.

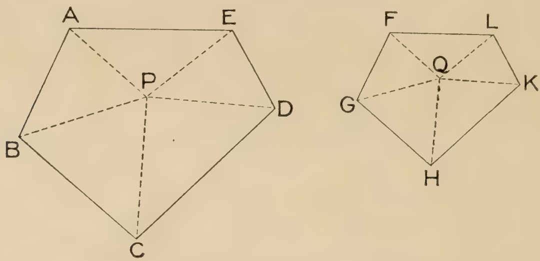
Show that FKLMN is similar to ABCDE.



4. Twice as many polygons may be described on a given line-segment GH, each similar to a given polygon, as the given polygon has sides.

196. **Problem :—To divide similar polygons into similar triangles.**

Let ABCDE, FGHKL be similar polygons of which AB and FG are corresponding sides.



It is required to divide ABCDE and FGHKL into similar △s.

Take any point P within the polygon ABCDE. Join PA, PB, PC, PD, PE.

Make ∠ GFQ = ∠ BAP and ∠ FGQ = ∠ ABP, and let the arms of these ∠s meet at Q.

Join QH, QK, QL.

∠ PAB = ∠ QFG and ∠ PBA = ∠ QGF; ∴ ∠ FQG = ∠ APB, and consequently △s ABP, FGQ are similar;

$$\therefore \frac{QG}{PB} = \frac{FG}{AB}.$$

But, by definition of similar polygons,

$$\frac{FG}{AB} = \frac{GH}{BC}.$$

$$\therefore \frac{QG}{PB} = \frac{GH}{BC}.$$

Also ∠ FGH = ∠ ABC and ∠ FGQ = ∠ ABP;

∴ ∠ QGH = ∠ PBC.

Then in $\triangle$s QGH, PBC, $\dfrac{QG}{PB} = \dfrac{GH}{BC}$, and $\angle QGH = \angle PBC$.

$\therefore$ these $\triangle$s are similar. (§ 112.)

In the same manner it may be shown that the remaining pairs of corresponding $\triangle$s are similar.

197. **Theorem:—The areas of similar polygons are proportional to the squares on corresponding sides.**

Using the diagram and construction of § 196,

It is required to show that $\dfrac{\text{polygon FGHKL}}{\text{polygon ABCDE}} = \dfrac{\text{sq. on FG}}{\text{sq. on AB}}$.

$\triangle$s FGQ, ABP are similar,

$$\therefore \frac{\triangle\ FGQ}{\triangle\ ABP} = \frac{\text{sq. on GQ}}{\text{sq. on BP}}. \qquad (\S\ 191.)$$

$$\text{Similarly } \frac{\triangle\ QGH}{\triangle\ PBC} = \frac{\text{sq. on GQ}}{\text{sq. on BP}}.$$

$$\therefore \frac{\triangle\ QGF}{\triangle\ PAB} = \frac{\triangle\ QGH}{\triangle\ PBC} = \text{(in the same manner)}$$

$$\frac{\triangle\ QHK}{\triangle\ PCD} = \frac{\triangle\ QKL}{\triangle\ PDE} = \frac{\triangle\ QLF}{\triangle\ PEA}.$$

But, if any number of fractions be equal to each other, the sum of their numerators divided by the sum of their denominators equals each of the fractions.

Now the sum of the numerators of the equal fractions is the polygon FGHKL, and the sum of the denominators is the polygon ABCDE;

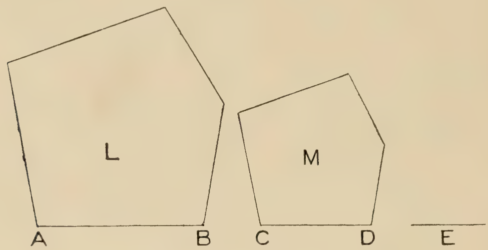
$$\therefore \frac{\text{polygon FGHKL}}{\text{polygon ABCDE}} = \frac{\triangle\ QFG}{\triangle\ PAB}.$$

$$\text{But } \frac{\triangle\ QFG}{\triangle\ PAB} = \frac{\text{sq. on FG}}{\text{sq. on AB}};$$

$$\therefore \frac{\text{polygon FGHKL}}{\text{polygon ABCDE}} = \frac{\text{sq. on FG}}{\text{sq. on AB}}.$$

198. **Theorem:—If three straight line-segments be in continued proportion, the first is to the third as any polygon on the first is to the similar and similarly described polygon on the second.**

Let AB, CD, E be three st. line-segments, such that AB : CD = CD : E, and L, M similar polygons having AB, CD corresponding sides.



It is required to show that polygon L : polygon M = AB : E.

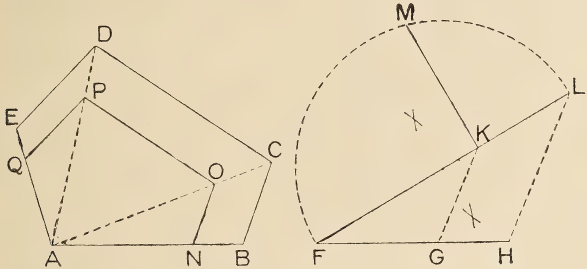
Let a, b, c be the numerical measures of AB, CD, E respectively.

Then $\frac{a}{b} = \frac{b}{c}$; and $\frac{\text{polygon L}}{\text{polygon M}} = \frac{a^2}{b^2}$ (§ 197.)

$$= \frac{a}{b} \times \frac{a}{b} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{c} = \frac{\text{AB}}{\text{E}}.$$

199. Problem:—To make a polygon similar to a given polygon and such that their areas are in a given ratio.

Let ABCDE be the given polygon and FG, GH two given line-segments.



It is required to make a polygon similar to ABCDE, and such that its area is to that of ABCDE as GH is to FG.

Find KL a fourth proportional to FG, GH, AB. (§ 105.)

Find KM a mean proportional to FK, KL. (§ 149.)

Cut off AN = KM, and on AN construct a polygon ANOPQ similar to ABCDE.

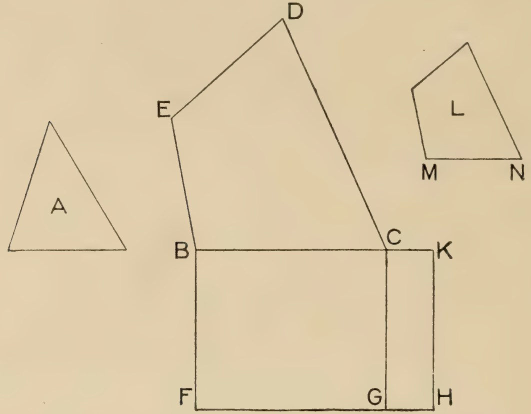
AB : AN = AN : KL, ∴ polygon ABCDE : polygon ANOPQ = AB : KL, (§ 198.)

= FG : GH

∴ polygon ANOPQ : polygon ABCDE = GH : FG.

200. **Problem:—To make a rectilineal figure equal to one given rectilineal figure and similar to another.**

Let A, BCDE be the two given figures.



It is required to make a figure equal in area to A and similar to BCDE.

On BC make a rect. BG equal to BCDE, and on GC make a rect. CH equal to A. Find MN a mean proportional between BC, CK. On MN describe a figure L similar to BCDE such that BC, MN are corresponding sides.

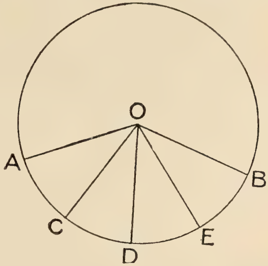
BC : MN = MN : CK, and ∴, by § 198,

$$\frac{\text{BCDE}}{\text{L}} = \frac{\text{BC}}{\text{CK}} = \frac{\text{rect. BG}}{\text{rect. CH}} = \frac{\text{BCDE}}{\text{A}}.$$

∴ L = A, and L was made similar to BCDE.

Arcs and Angles.

201. Suppose an angle AOB at the centre of a circle to be divided into a number of equal parts AOC, COD, DOE, EOB.

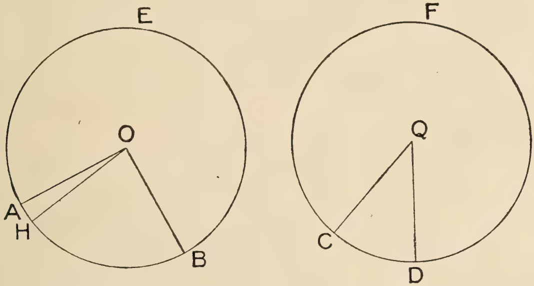


Then, by § 154, the arcs AC, CD, DE, EB are equal to each other, and whatever multiple the angle AOB is of the angle AOC, the arc AB is the same multiple of the arc AC.

Thus, if an angle at the centre of a circle be divided into degrees and contain a of them, the arc subtending the angle will contain the arc subtending one degree a times.

202. **Theorem:—In equal circles, or in the same circle, angles at the centres are as the arcs on which they stand.**

Let AOB, CQD be two $\angle$s at the centres of the equal circles ABE, CDF.



It is required to show that $\angle$ AOB : $\angle$ CQD = arc AB : arc CD.

Let $\angle$s AOB, CQD contain a unit $\angle$ AOH a, b times respectively; then arcs AB, CD contain arc AH a, b times respectively.

$$\therefore \frac{\angle AOB}{\angle AOC} = \frac{a \times \angle AOH}{b \times \angle AOH} = \frac{a}{b} = \frac{a \times \text{arc AH}}{b \times \text{arc AH}} = \frac{\text{arc AB}}{\text{arc CD}}.$$

203.—Exercises.

1. In equal circles, or in the same circle, angles at the circumferences are as the arcs on which they stand.

2. In equal circles, or in the same circle, the areas of sectors are as their angles.

COURSE IN GEOMETRY FOR HIGH SCHOOLS.

The following details of the Course in Geometry prescribed for the Lower and Middle Schools of the High Schools and Collegiate Institutes of Ontario are taken from Appendix C of the Regulations, and are given here for the information of teachers and pupils.

The references are to sections and examples of Parts I. and II. of this book.

The first thirteen of the constructions and the first nineteen of the theorems are prescribed for candidates for District teachers' non-professional certificates, in addition to the Practical Geometry of the Lower School.

A.—Constructions.

To construct a triangle with sides of given lengths. Pt. I., § 33, Ex. 5 ; Pt. II., § 52, Ex. 6.

To construct an angle equal to a given rectilineal angle. Pt. I., § 43 ; Pt. II., § 52, Ex. 5.

To bisect a given angle. Pt. I., § 41 ; Pt. II., § 52, Ex. 1.

To bisect a given straight line. Pt. I., § 63 ; Pt. II., § 52, Ex. 2.

To draw a line perpendicular to a given line from a given point in it. Pt. I., § 49 ; Pt. II., § 52, Ex. 3.

To draw a line perpendicular to a given line from a given point not in the line. Pt. I., §§ 65, 66 ; Pt. II., § 52, Ex. 4.

Locus of a point equidistant from two given lines. Pt. II., § 64, Ex. 2.

Locus of a point equidistant from two given points. Pt. II., § 132.

To draw a line parallel to another, through a given point. Pt. I., §§ 97, 100; Pt. II., § 57, Ex. 9 and 10.

To divide a given line into any number of equal parts. Pt. I., § 163, § 164, Ex. 7; Pt. II., § 104.

To describe a parallelogram equal to a given triangle, and having an angle equal to a given angle. Pt. II., § 80.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle. Pt. II., § 82, Ex. 8; § 83.

On a given straight line to describe a parallelogram equal to a given triangle, and having an angle equal to a given angle. Pt. II., § 84.

To find the centre of a given circle. Pt. II., § 140, Ex. 2.

From a given point to draw a tangent to a given circle. Pt. I., § 143; Pt. II., § 160.

On a given straight line to construct a segment of a circle containing an angle equal to a given angle. Pt. I., § 138, Exs. 10 to 14; Pt. II., § 164, Ex. 5.

From a given circle to cut off a segment containing an angle equal to a given angle. Pt. II., § 164, Ex. 3.

In a circle to inscribe a triangle equiangular to a given triangle. Pt. II., § 164, Ex. 4.

To find locus of centres of circles touching two given lines. Pt. II., § 159, Ex. 7 and 9; § 180, Ex. 1.

To inscribe a circle in a given triangle. Pt. I., § 145; Pt. II., § 177.

To describe a circle touching three given straight lines. Pt. II., § 178; § 180, Ex. 14.

To describe a circle about a given triangle. Pt. I., § 64, Exs. 3 to 7; Pt. II., § 140, Ex. 1.

About a given circle to describe a triangle equiangular to a given triangle. Pt. II., § 181.

To divide a given line similarly to another given divided line. Pt. II., § 106 ; § 107, Ex. 6.

To find the fourth proportional to three given lines. Pt. II., § 105.

To describe a polygon similar to a given polygon, and with the corresponding sides in a given ratio. Pt. II., § 194 ; § 195, Ex. 2.

To find the mean proportional between two given straight lines. Pt. I., § 169 ; Pt. II., § 149.

To construct a polygon similar to a given polygon, and such that their areas are in a given ratio. Pt. II., § 199.

To describe a polygon of given shape and size. Pt. II., § 200.

B.—Theorems.

The sum of the angles of any triangle is equal to two right angles. Pt. I., § 72 ; Pt. II., § 61.

The angles at the base of an isosceles triangle are equal, with converse. Pt. II., § 34 ; § 38.

If the three sides of one triangle be equal respectively, to the three sides of another, the triangles are equal in all respects. Pt. II., § 49.

If two sides and the included angle of one triangle be equal to two sides and the included angle of another triangle, the triangles are equal in all respects. Pt. II., § 30.

If two angles and one side of triangle be equal to two angles and the corresponding side of another, the triangles are equal in all respects. Pt. II., § 63.

If two sides and an angle opposite one of these sides be equal respectively in two triangles, the angles opposite the other pair of equal sides are either equal or supplemental. Pt. II., § 65.

The sum of the exterior angles of a polygon is four right angles, Pt. II., § 62, Ex. 16.

The greater side of any triangle has the greater angle opposite it. Pt. II., § 37.

The greater angle of any triangle has the greater side opposite it. Pt. II., § 42.

If two sides of one triangle be equal respectively to two sides of another, that with the greater contained angle has the greater base, with converse. Pt. II., § 48, § 50.

If a transversal fall on two parallel lines, relations between angles formed, with converse. Pt. II., §§ 56, 57, Ex. 2, 3; §§ 59, 60, Ex. 1, 2.

Lines which join equal and parallel lines towards the same parts are themselves equal and parallel. Pt. II., § 60, Ex. 5.

The opposite sides and angles of a parallelogram are equal and the diagonal bisects it. Pt. II., § 64, Ex. 7, 8, 9.

Parallelograms on the same base, or on equal bases, and between the same parallels, are equal. Pt. II., § 76, Ex. 1.

Triangles on the same base, or on equal bases, and between the same parallels, are equal. Pt. II., § 76, Ex. 2.

Triangles equal in area, and on the same base, are between the same parallels. Pt. II., § 77.

If a parallelogram and a triangle be on the same base, and between the same parallels, the parallelogram is double the triangle. Pt. II., § 76, Ex. 3.

Expressions for area of a parallelogram, and area of a triangle. Pt. II., § 74.

The compliments of the parallelograms about the diagonal of any parallelogram are equal. Pt. II., § 79.

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the sides. Pt. II., § 121; § 128, Ex. 13.

If a straight line be divided into any two parts, the sum of the squares on the parts, together with twice the rectangle contained by the parts, is equal to the square on the whole line. Pt. II., § 86.

The square on a side of any triangle is equal to the sum of the squares on the two other sides — twice the rectangle contained by either of these sides and the projection of the other side on it. Pt. II., § 127 ; § 128, Ex. 1.

If more than two equal straight lines can be drawn from the circumference of a circle to a point within it, that point is the centre. Pt. II., § 137.

The diameter is the greatest chord in a circle, and a chord nearer the centre is greater than one more remote. Also the greater chord is nearer the centre than the less. Pt. II., § 139 ; § 140, Ex. 4 ; § 140, Ex. 7.

The angle at the centre of a circle is double the angle at the circumference on the same arc. Pt. II., § 141.

The angles in the same segment of a circle are equal, with converse. Pt. II., § 146 ; § 147.

The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles, with converse. Part II., § 151 ; § 152.

The angle in a semi-circle is a right angle ; in a segment greater than a semi-circle less than a right angle ; in a segment less than a semi-circle greater than a right angle. Pt. II., § 145 ; § 148, Ex. 7.

A tangent is perpendicular to the radius to the point of contact ; only one tangent can be drawn at a given point ; the perpendicular to the tangent at the point of contact passes through the centre ; the perpendicular from centre on tangent passes through the point of contact. Pt. II., § 157 ; § 158 ; § 159, Ex. 3.

If two circles touch, the line joining the centres passes through the point of contact. Pt. II., § 172.

The angles which a chord drawn from the point of contact makes with the tangent are equal to the angles in the alternate segments. Pt. II., § 163.

The rectangles under the segments of intersecting chords are equal. Pt. II., § 165.

If $OA.OB = OC^2$, OC is a tangent to the circle through A, B and C. Pt. II., § 168.

Triangles of the same altitude are as their bases. Pt. II., § 100.

A line parallel to the base of a triangle divides the sides proportionally, with the converse. Pt. II., § 102 ; § 103, Ex. 3.

If the verticle angle of a triangle be bisected, the bisector divides the base into segments that are as the sides, with converse. Pt. II., § 114, § 115.

The analogous proposition when the exterior angle at the vertex is bisected, with converse. Pt. II., § 116, Ex. 3, 4.

If two triangles are equiangular, the sides are proportional. Pt. II., § 109.

If the sides of two triangles are proportional, the triangles are equiangular. Part II., § 110.

If the sides of two triangles about equal angles are proportional, the triangles are equiangular. Pt. II., § 112.

If two triangles have an angle in each equal, and the sides about two other angles proportional, the remaining angles are equal or supplemental. Pt. II., § 113.

Similar triangles are as the squares on corresponding sides. Pt. II., § 191.

The perpendicular from a right angle of a right-angled triangle on the hypotenuse divides the triangle into two which are similar to the original triangle. Pt. II., § 119.

In equal circles, angles, whether at the centres or circumferences, are proportional to the arcs on which they stand. Pt. II., § 202 ; § 203, Ex. 1.

The areas of two similar polygons are as the squares on corresponding sides. Pt. II., § 197.

If three lines be proportional, the first is to the third as the figure on the first to the similar figure on the second. Pt. II., § 198.

Questions and easy deductions on the preceding constructions and theorems.

NOTE.—In the formal deductive Geometry modifications of Euclid's treatment of the subject will be allowed, though not required, as follows :—

The employment of the "hypothetical construction."

The free employment of the method of superposition, including the rotation of figures about an axis, or about a point in a plane.

A modification of Euclid's parallel postulate.

A treatment of ratio and proportion restricted to the case in which the compared magnitudes are commensurable.